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OMEGA SYSTEM AVAILABILITY AS A GLOBAL MEASURE OF NAVIGATION ACCURACY

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September 1990
FINAL REPORT

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16. ABSTRACT

An augmented Omega System Availability model is presented in which system availability is treated as the probability that navigation accuracy is better than a selected threshold. In addition to the four sub-models used in connection with the previously developed System Availability Model, two new sub-models are presented which are needed for a probabilistic description of navigation accuracy as a measure of Omega system availability. One new sub-model describes the sources and statistical distribution of Omega phase errors, while the other sub-model defines the transformation of phase error to position error based on a generic method of Omega navigation commonly used in conventional airborne Omega navigation receiver systems. Resources required for the calculation of the system availability index based on the augmented model are estimated. Finally, as an alternative to the probabilistic model, a very simple GDOP-type form is proposed which could be included as a collective signal criterion in the current deterministic version of the PACE workstation.

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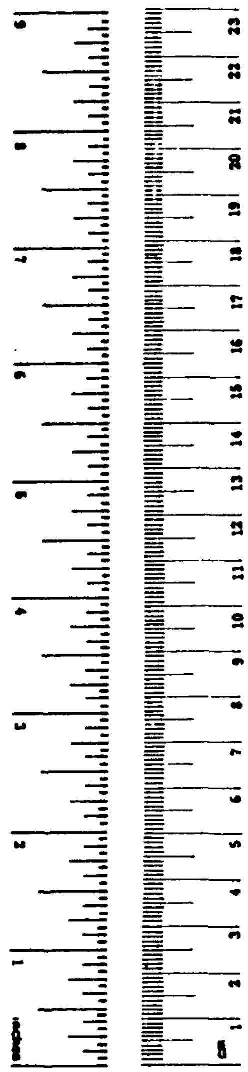
METRIC CONVERSION FACTORS

Approximate Conversions to Metric Measures

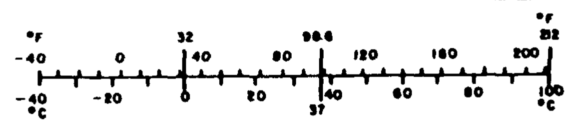
Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
in	inches	*2.5	centimeters	cm
ft	feet	30	centimeters	cm
yd	yards	0.9	meters	m
mi	miles	1.6	kilometers	km
		AREA		
in^2	square inches	6.5	square centimeters	cm^2
ft^2	square feet	0.09	square meters	m^2
yd^2	square yards	0.8	square meters	m^2
mi^2	square miles	2.6	square kilometers	km^2
	acres	0.4	hectares	ha
		MASS (weight)		
oz	ounces	28	grams	g
lb	pounds	0.45	kilograms	kg
	short tons (2000 lb)	0.9	tonnes	t
		VOLUME		
tsp	teaspoons	5	milliliters	ml
Tbsp	tablespoons	15	milliliters	ml
fl oz	fluid ounces	30	milliliters	ml
c	cups	0.24	liters	l
pt	pints	0.47	liters	l
qt	quarts	0.95	liters	l
gal	gallons	3.8	liters	l
ft^3	cubic feet	0.03	cubic meters	m^3
yd^3	cubic yards	0.76	cubic meters	m^3
		TEMPERATURE (exact)		
°F	Fahrenheit temperature	5/9 (after subtracting 32)	Celsius temperature	°C

*1 in = 2.54 (exactly). For other exact conversions and more detailed tables, see NBS Misc. Publ. 286, Units of Weights and Measures, Price $2.25, SD Catalog No. C13.10:286.



Approximate Conversions from Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
mm	millimeters	0.04	inches	in
cm	centimeters	0.4	inches	in
m	meters	3.3	feet	ft
m	meters	1.1	yards	yd
km	kilometers	0.6	miles	mi
		AREA		
cm^2	square centimeters	0.16	square inches	in^2
m^2	square meters	1.2	square yards	yd^2
km^2	square kilometers	0.4	square miles	mi^2
ha	hectares (10,000 m^2)	2.5	acres	
		MASS (weight)		
g	grams	0.035	ounces	oz
kg	kilograms	2.2	pounds	lb
t	tonnes (1000 kg)	1.1	short tons	
		VOLUME		
ml	milliliters	0.03	fluid ounces	fl oz
l	liters	2.1	pints	pt
l	liters	1.06	quarts	qt
l	liters	0.26	gallons	gal
m^3	cubic meters	35	cubic feet	ft^3
m^3	cubic meters	1.3	cubic yards	yd^3
		TEMPERATURE (exact)		
°C	Celsius temperature	9/5 (then add 32)	Fahrenheit temperature	°F



NOTICE

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The author wishes to acknowledge the contribution of Captain R.J. Wenzel, Commanding Officer, U.S. Coast Guard Omega Navigation System Center to the development of the phase error sub-model of the overall system availability model. In particular, Captain Wenzel discovered the predominance of the day-to-day Omega signal phase variation over the phase error induced by the propagated noise in the receiver's front-end bandwidth. This discovery had a profound effect on the development and structure of the phase error sub-model presented in this Technical Information Memorandum.

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1. INTRODUCTION

1.1 ORIGINAL AND ENHANCED SYSTEM AVAILABILITY MODELS

The System Availability Model was developed to provide an overall measure, or index, of Omega system performance by combining performance measures from the following four sub-models:

- Omega receiver system reliability/availability
- Omega station reliability/availability
- Omega signal coverage (spatial/temporal)
- Omega user regional priority.

The model structure consists of a probabilistic definition of the system availability index (P_{SA}) in terms of probabilistic/deterministic structures for the above four sub-models. The initial development of this model (Ref. 1) treated the first two elements above probabilistically, while the third and fourth were analyzed using deterministic sub-models. In subsequent work (Ref. 2), the model was enhanced to treat certain signal coverage parameters (signal amplitude and noise level) as random variables.

Flexibility is built into the System Availability Model by allowing sub-models to be "turned on and off" as desired. For example, in the first sub-model, the probability that the receiver system is reliable/available can be set equal to one, thus eliminating any influence of receiver system reliability on P_{SA}. Similarly, the contribution of the fourth element above to P_{SA} can be eliminated by setting all regional priorities (weightings) equal. The system availability index, computed on a monthly basis, can be monitored as a continuing measure of system performance. The index can also be used to compare the effects of system options, e.g., reductions in radiated power at one or more stations.

The system availability index, P_{SA}, was originally defined (Ref. 1) as the probability that, at any time and at any point on the earth's surface, an Omega user's receiver is properly functioning and three or more usable Omega signals are available to permit successful navigation, position-fixing, or other use of the system. With this definition and the fact that receiver reliability/availability is independent of station signal access, P_{SA} may be expressed as:

$$P_{SA} = P_R \, P_A$$

where P_R is the probability that the receiver is reliable/available and P_A is the probability that three or more usable signals (in space) are available.

The phrases "any time" and "any point" in the above definition mean that the probability is computed over all spatial and temporal variables. In practice, the two-dimensional spatial unit (cell) is taken to be about 10 deg. (latitude) by 10 deg. (longitude) in size. Since coverage at a fixed UT hour is assumed constant over the days in a month, the appropriate temporal variables are UT hour (1-24) and month. Although signal coverage is specified for only four months (February, May, August, and November), the monthly change in signal coverage is relatively small, so that interpolation of signal coverage parameters over months is permissible. As a result, coverage information for all 12 months can be derived (Ref. 2). To be strictly compatible with the above definition, P_{SA} is computed over all cells on the globe and over all hours and months in a year. However, it is also useful to compute P_{SA} for various ranges of the temporal variables, e.g.:

- Fixed hour/month
- Fixed month/average over 24 UT hours
- Fixed month/maximize over 24 UT hours
- Fixed month/minimize over 24 UT hours
- Maximize or minimize over all months and hours.

The access probability, P_A, is the probability that three or more usable Omega signals are accessible at any point (or a given point) in time and space. The threshold of three signals stems from conventional Omega usage and is not a limitation of the model which permits an arbitrary minimum usable number of signals. In developing the model (Ref. 1), P_A is written as a sum of two-factor terms — the first factor known as the coverage element and the second factor called the network reliability factor. Most of P_{SA}'s spatial and temporal dependence is contained in the coverage elements (spatial dependence may also be found in the user regional priority weightings). A separate month and year dependence for P_{SA} arises from the network reliability factors. Coverage elements define coverage in terms of the following signal properties:

- Signal-to-noise ratio (SNR) in a receiver's "front-end" bandwidth
- Relative strength and phasing of signal modes comprising the total signal
- Ratio of long-path to short-path signal amplitude
- Path/terminator crossing angle.

Criteria imposed on the above signal properties to determine signal coverage are discussed in Ref. 1. Network reliability factors define the probability that each station of the Omega transmitting network is in a specific on-air/off-air condition. The station reliability sub-model includes three types of off-air conditions:

- Unscheduled off-air
- Scheduled off-air (excluding annual maintenance)
- Scheduled annual maintenance.

Recent-year station reliability data were used to determine average durations for the above off-air conditions. From these data, network reliability factors are derived using operational constraints governing concurrent off-air conditions.

As mentioned above, the enhanced system availability model (Ref. 2) treats the received signal amplitude and noise level as random variables. This means that these values are not precisely determined at a given point in space (latitude/longitude) and time (hour/month) but the parameters of their statistical distribution (lognormal in form) are given by an augmented version of the coverage database. Whereas the original database supplies fixed, deterministic values of signal amplitude and noise level for a given space/time point, the enhanced model treats these as mean values and requires additional standard deviation information from the database. Thus, the coverage database is expanded to include algorithms for computing the signal amplitude and noise level standard deviations. A much smaller database is needed to extract statistical data on station reliability.

Since P_{SA} is expected to be a useful tool for evaluating and comparing system planning options (e.g., reducing station power or eliminating stations), calculation of P_{SA} has been imbedded in a workstation system known as Performance Assessment and Coverage Evaluation (PACE). This system is designed for easy selection and display of scenario inputs, including time (hours/months) and space (entire globe, specific region, or disjoint collection of cells) and provides a split-screen display for comparing system options. The system computes P_{SA} as defined above (globally, averaged over all times) or over restricted domains in space and time. With respect to annual station maintenance/off-air periods, PACE provides an option of computing (for a given month) P_{SA} including or excluding the station undergoing annual maintenance. Strictly speaking, P_{SA} does not apply to time periods less than one month (although it may be restricted to a fixed hour for each day in the month), since the *a priori* scheduled and

unscheduled off-air probabilities must be defined over a sufficiently long time interval (see Appendix A of Ref. 1). However, for a number of important applications (such as transoceanic aircraft flights), the period of time over which computation of P_{SA} is required is short compared to a typical annual maintenance off-air period. Thus, a monthly measure of P_{SA} is not indicative of system performance for these applications. Consequently, options are included in PACE to compute P_{SA} (for a given month) under three conditions:

- The station assigned annual maintenance for the month is off-air
- The station assigned annual maintenance for the month is not currently off-air for annual maintenance but has specific unscheduled and scheduled off-air probabilities
- The station assigned annual maintenance for the month has a status which is defined probabilistically.

PACE also contains options for computing P_{SA} with probabilistic or deterministic definitions of signal amplitude and noise level. For the machine on which the PACE workstation is hosted, the probabilistic calculation is lengthy and better suited for off-line computations of P_{SA} at a single point in space and time. The deterministic calculation is rapid, however, so that global/multiple-time computations are feasible for operational use of the workstation.

1.2 OBJECTIVE

The objective of this report is to explore the possibility of redefining the Omega System Availability Model/Index in terms of position/navigation accuracy. This redefinition represents a shift in the system performance criteria from maximizing signal "coverage" (number of usable station signals available, independent of their relative contributions to position accuracy) to maximizing position/navigation accuracy over time and space. This redefined index is also consistent with performance measures for other systems (e.g., NAVSTAR GPS) which define coverage in terms of relative position accuracy (dilution of precision).

The new index (henceforth referred to as P_{SAA}) should preserve the probabilistic character of the original System Availability Index to provide a meaningful indicator of system performance at the user level. This means that the specified position/navigation accuracy should be consistent with the most widely representative Omega receiver/processor systems in current use. To be useful to the system manager/operator, however, the index should be sufficiently

sensitive to reflect substantial reductions/improvements in system accuracy over important global regions. The feasibility of incorporating P_{SAA} into PACE should also be determined and, if necessary, simpler measures of system accuracy should be proposed.

The various measures of navigation/position accuracy which appear in the literature are often misunderstood and misused. These accuracy measures can be obtained from the position error probability density function which is derived in the development of an analytic expression for P_{SAA}. The proper interpretation of these measures should be quantitatively described.

The technical discussion presented in Chapter 2 shows that P_{SAA} can meet the above goals. In addition, the new index does not require modification of the sub-models making up the original System Availability Model, but it does require additional sub-models for phase errors and navigation/positioning procedures. Thus, the expanded definition of P_{SA} specifies not only the probability that usable signals are accessible, but that, collectively, they provide a given position/navigation accuracy.

1.3 APPROACH

Redefinition of the System Availability Index requires that additional sub-models be developed, not to improve the fidelity of the original model but to calculate the new quantities introduced in the redefined index (P_{SAA}). Since P_{SAA} defines position/navigation error, additional sub-models are required for both signal phase error and position estimation. Phase errors observed at Omega/VLF frequencies are characterized by random and bias components of similar magnitude. Thus, both components must be included in the phase error sub-model. The transformation of received Omega signal phase to position is strongly dependent on receiver mechanization. Error models involving this transformation frequently assume a hyperbolic mechanization in which two or more phase differences are processed. Most Omega receiver/processors currently employ a range-only mechanization using a navigation filter which processes multiple, redundant phase measurements. Although receiver mechanization and navigation filter designs are manufacturer-specific, the approach taken here is to adopt a simple model of position estimation which is reasonably representative of current Omega receiver/processor implementations.

The two new sub-models identified above, together with the previous sub-models, constitute the augmented system availability model which is used to develop the analytical form for

P_{SAA}. Since the position error density function is obtained in the process of deriving P_{SAA}, the development digresses somewhat to describe the error measures obtainable from this function. This is useful in relating the development herein to other analyses of position error.

Because of the complex dependence of the model on the system quantities, an alternative simple measure of position/navigation accuracy is also suggested. This method could be readily incorporated into a future version of PACE, if the rigorous model for P_{SAA} proves unwieldy.

1.4 REPORT OVERVIEW

Development of the augmented system availability model/algorithm is entirely contained in Chapter 2. Previously developed sub-models, e.g., the station reliability/availability sub-model, are reviewed, followed by development of the new sub-models. The analytical structure of P_{SAA} is then established and the forms of the probability functions are developed by applying the sub-models. Resources (processing time/memory) required for calculation of P_{SAA} are estimated and a simplified accuracy measure for possible inclusion into PACE operation is briefly described.

Chapter 3 provides a summary of the development, including general conclusions and recommendations. Mathematical details of the phase error/position estimation error transformation are included in the Appendices.

2. AN AUGMENTED SYSTEM AVAILABILITY MODEL INCORPORATING POSITION/NAVIGATION ACCURACY

In this chapter, the original system availability model/algorithm is augmented to include position/navigation accuracy. Previously developed sub-models of Omega receiver and station reliability/availability sub-models, also needed for the augmented model, are reviewed in Section 2.1. In Section 2.2, new sub-models are developed for phase error and the transformation to position estimation error. The analytical structure of the redefined system availability index, P_{SAA}, is presented in Section 2.3 in terms of the position error probability distribution function and probabilistic results from each of the sub-models. An estimate of the relative amount of computational resource needed to calculate P_{SAA} is contained in Section 2.4. This section also describes a simplified measure of accuracy, as an alternative to P_{SAA}, which could be readily incorporated into the Performance Assessment and Coverage Evaluation (PACE) workstation.

2.1 PREVIOUSLY DEVELOPED SUB-MODELS USED IN THE AUGMENTED MOD.‍L

For the augmented system availability model, the system availability index, P_{SAA}, addresses the probability of obtaining a desired position/navigation accuracy. Clearly the even that a given accuracy is obtained depends upon a series of other events, including proper operation of the user's receiver, transmitting station(s) on-air, and accessibility of usable signals. Models governing these events have been developed previously and their structure is reviewed in the following subsections.

2.1.1 Receiver Reliability/Availability Sub-model

The probability that a receiver is functioning properly, P_R, can be expressed in terms of the receiver's MTBF (mean time between failure) and the MTTR (mean time to repair) as explained in the original system availability model (Ref. 1). The MTBF and MTTR are approximately the same for receivers in a given receiver *class*, e.g., civil aircraft, military marine, or meteorological balloon systems. Although P_R clearly depends on the receiver class, other sub-models also depend on receiver class through the coverage elements (described in Section 2.1.3) which are functions of the criteria for assigning coverage. One of the criteria

involves signal-to-noise ratio (SNR), which has a threshold specified as input but often linked to receiver sensitivity and other parameters. Since these receiver parameters are roughly the same for a given receiver class, the SNR criterion may be keyed to receiver class.

Applications of the augmented system availability model are generally not expected to involve multiple receiver classes so that, throughout this report, the receiver reliability/availability (P_R) is assumed constant, independent of receiver class i. Moreover, since there is no meaningful average reliability/availability figure to describe all receiver classes, P_R will be set to 1 for most applications of this model. It is important to recognize, however, that the reliability/availability sub-model can be invoked at any time with little or no change to the other sub-models.

2.1.2 Station Reliability/Availability Sub-model

In the original system availability model (Ref. 1), station off-air events are treated as random variables, both in terms of event onset time and duration. The station reliability/availability sub-model defines two types of off-airs: unscheduled and scheduled. Unscheduled off-airs occur as a result of unforeseen circumstances — usually equipment failures. At the beginning of a month, the occurrence probability of an unscheduled off-air is essentially uniform over the month although compilations of monthly total off-air statistics are available as a function of station. Scheduled off-airs are planned conditions under which signal generation temporarily ceases. An important class of these events is the annual maintenance off-air for each station. A station's annual maintenance occurs in a specific month, unique to that station, and generally includes routine maintenance/repair work. Since users are usually notified of these annual maintenance periods well in advance, the occurrence time and duration of these events may be considered deterministic. Advance notice of other types of scheduled off-airs (a few days to two weeks) is such that these events may be considered random to a user at the beginning of a month (basic time interval over which off-air probabilities are defined). Despite its randomness, this type of scheduled off-air differs from an unscheduled off-air in an important way, as noted below.

The station reliability sub-model defines certain relationships between unscheduled and scheduled off-air events at the same or different stations based on intrinsic definitions and operational practice. The occurrence of an unscheduled off-air at a station is independent of the occurrence of an unscheduled or scheduled off-air event at any other station. However, unscheduled and scheduled off-air events at the same station are exclusive, i.e., they cannot occur at the same time. Because of Omega system operational/management policy, scheduled

off-air events at different stations are interdependent in the sense that they are forbidden from simultaneous occurrence.

For PACE, it is assumed that the unscheduled off-air probability is 0.001 for all stations and months. Using the notation introduced in developing the system availability model (Ref. 1), this requirement is expressed

$$P(\bar{T}_i^u) = 0.001 \quad ; \; i = 1, 2, \ldots, 8$$

where $\bar{T}_i^u$ is the event that station i is in an unscheduled off-air condition. The scheduled off-air (excluding annual maintenance) event probabilities are assumed to be station-specific but are constant for each month over the year (all off-air probabilities are assumed independent of year). These values are obtained by averaging observed scheduled off-air times (excluding annual maintenance) over three recent years (data obtained from Ref. 3) for each station. Scheduled off-air probabilities for annual maintenance are computed by averaging the off-air times for each station's maintenance month over three recent years (Ref. 3). The resulting data are shown in Table 2.1-1. In this table, the first entry (for a given month/station combination) is the fixed unscheduled off-air event probability, the second is the scheduled off-air (excluding annual maintenance) event probability, and the third is the scheduled annual maintenance event probability. Note that the scheduled off-air (excluding annual maintenance) event probability is specified even for the months corresponding to a station's annual maintenance. This is because a scheduled off-air event (with a few days advance notice) may occur during the month, before or after the annual maintenance period with approximately the same relative probability as during the other months. Unscheduled off-air events at one or more stations may also occur, but scheduled events differ probabilistically in that they are *never* concurrent among all stations.

2.1.3 Signal Coverage Sub-model

In descriptions of Omega signal usage, the word "coverage" is defined in terms of the usability of a single signal for position/navigation. Although the usability is broken down into categories (see below), the essential idea is that the signal phase must be an approximately linear, regularly varying, increasing function of distance from a transmitting station (for a fixed time/time interval). In some past work (Ref. 4), the notion of coverage has been extended to multiple station signals, e.g., by inclusion of geometric dilution of precision (GDOP) thresholds under certain conditions (3-4 station signals in hyperbolic mode). In general, however, and in this report, the word "coverage" refers to a single station signal.

Table 2.1-1 Station Reliability/Availability Parameters for PACE*

	A	B	C	D	E	F	G	H
JAN	.00100 .00269 .00000	.00100 .00037 .00000	.00100 .03604 .00000	.00100 .00024 .00000	.00100 .00163 .00000	.00100 .00030 .00000	.00100 .00061 .00000	.00100 .00005 .00000
FEB	.00100 .00269 .00000	.00100 .00037 .34569	.00100 .03604 .00000	.00100 .00024 .00000	.00100 .00163 .00000	.00100 .00030 .00000	.00100 .00061 .00000	.00100 .00005 .00000
MAR	.00100 .00269 .00000	.00100 .00037 .00000	.00100 .03604 .00000	.00100 .00024 .00000	.00100 .00163 .00000	.00100 .00030 .20511	.00100 .00061 .00000	.00100 .00005 .00000
APR	.00100 .00269 .00000	.00100 .00037 .00000	.00100 .03604 .00000	.00100 .00024 .00000	.00100 .00163 .00000	.00100 .00030 .00000	.00100 .00061 .00000	.00100 .00005 .00000
MAY	.00100 .00269 .00000	.00100 .00037 .00000	.00100 .03604 .00000	.00100 .00024 .00000	.00100 .00163 .00000	.00100 .00030 .00000	.00100 .00061 .00000	.00100 .00005 .00000
JUN	.00100 .00269 .00000	.00100 .00037 .00000	.00100 .03604 .28628	.00100 .00024 .00000	.00100 .00163 .00000	.00100 .00030 .00000	.00100 .00061 .00000	.00100 .00005 .00000
JUL	.00100 .00269 .00000	.00100 .00037 .00000	.00100 .03604 .00000	.00100 .00024 .07895	.00100 .00163 .00000	.00100 .00030 .00000	.00100 .00061 .00000	.00100 .00005 .00000
AUG	.00100 .00269 .11057	.00100 .00037 .00000	.00100 .03604 .00000	.00100 .00024 .00000	.00100 .00163 .00000	.00100 .00030 .00000	.00100 .00061 .00000	.00100 .00005 .00000
SEP	.00100 .00269 .00000	.00100 .00037 .00000	.00100 .03604 .00000	.00100 .00024 .00000	.00100 .00163 .61490	.00100 .00030 .00000	.00100 .00061 .00000	.00100 .00005 .00000
OCT	.00100 .00269 .00000	.00100 .00037 .00000	.00100 .03604 .00000	.00100 .00024 .00000	.00100 .00163 .00000	.00100 .00030 .00000	.00100 .00061 .00000	.00100 .00005 .32515
NOV	.00100 .00269 .00000	.00100 .00037 .00000	.00100 .03604 .00000	.00100 .00024 .00000	.00100 .00163 .00000	.00100 .00030 .00000	.00100 .00061 .01726	.00100 .00005 .00000
DEC	.00100 .00269 .00000	.00100 .00037 .00000	.00100 .03604 .00000	.00100 .00024 .00000	.00100 .00163 .00000	.00100 .00030 .00000	.00100 .00061 .00000	.00100 .00005 .00000

* First entry is the unscheduled off-air event probability; second entry is the scheduled off-air (excluding annual maintenance) event probability; third entry is the scheduled annual maintenance off-air event probability.

The signal coverage database is generated in two stages. In the first stage, signal amplitude and phase are computed (from theoretical models) for both the total (Mode-sum) signal and its Mode 1 component. These two components are separately specified because the Mode 1 component is assumed to adequately represent the total signal for navigation purposes, although the theoretically computed Mode-sum signal is actually received. The first-stage database contains the signal amplitude for the short-path/long-path (shorter/longer of the two great-circle paths between a transmitter and receiver) and phase for the short-path only. The amplitude for the short- and long-paths is separately specified because most navigation models assume that the received signal is propagated via the short-path whereas in some cases the long-path signal may actually dominate the total received signal amplitude. The data are specified for:

- Signal frequencies of 10.2 and 13.6 kHz
- Each of the eight Omega stations
- Radial paths at bearing intervals of approximately 10 degrees from each station
- Distance intervals of 500 km along each path
- Each of the 24 UT hours
- The months of February, May, August, and November.

In the second stage of the database generation, the signal parameters defined above are interpolated from the station radial path-based grid to a coarser-scale matrix/cell arrangement specified in Table 2.1-2. The noise data (median level and standard deviation) are extracted from the WGL/NRL model/algorithm (Ref. 5) for the specified cells and times. Both the phase deviation of the total signal phase from the Mode 1 phase and the dominant mode number are computed from the data in the first-stage database to provide quantitative information on modal interference. The path/terminator angle is also computed to test for conditions leading to a possible cycle slip/jump.

In the original system availability model (Ref. 1), the signal amplitude from the Omega signal coverage database and the noise level from the WGL/NRL noise model are treated as deterministic quantities. As explained above, the enhanced version of the system availability model treats signal amplitude and noise level as random quantities. Specifically, signal amplitude and noise level are assumed to be lognormally distributed (Ref. 2) with mean values obtained, respectively, from the signal coverage database and the WGL/NRL model/algorithm. Standard deviations for the signal amplitude distribution are available from an algorithm

Table 2.1-2 Latitude/Longitude Dimensions of Cells in Grid Structure for Signal Coverage Database (Matrix Format)

LATITUDE RANGE*	LATITUDE DIMENSION OF CELL	LONGITUDE DIMENSION OF CELL	NUMBER OF CELLS IN BOTH HEMISPHERES
0° to 40°	10°	10°	288
40° to 60°	10°	15°	96
60° to 75°	15°	15°	48
75° to 90°	15°	60°	12
TOTAL NUMBER OF CELLS = 444			

*Same for northern and southern hemisphere

described in Ref. 2. Standard deviations for the noise level distribution are obtained from the WGL/NRL model/algorithm mentioned above.

Data from the database/models described above do not solely determine coverage. Coverage access criteria, supplied as defaults or user input, are applied to the data to determine the usability of the given signal in the presence of the given noise. In the case of assumed deterministic coverage variables (e.g., path/terminator angle), application of an access criterion yields a "yes" (presence of coverage) or a "no" (absence of coverage). For assumed random coverage variables (e.g., noise level), the criterion furnishes a limit for a probability distribution function so that satisfaction of the criterion is determined in a probabilistic sense. Default coverage access criteria are given as follows:

- Phase deviation (angle of phasor difference between Mode 1 signal and total signal) must be less than 20 centicycles
- Dominant mode must be Mode 1
- Ratio of total long-path signal amplitude to total short-path signal amplitude must be less than -3 dB
- Ratio of total short-path signal amplitude (assumed mean value) to median noise level (assumed mean value) in a 100 Hz BW must be greater than -20 dB
- Angle between the (great-circle) propagation path and the (great-circle) terminator must be greater than 5 degrees.

Signals which satisfy the deterministic coverage access criteria for a given scenario are said to comprise the *maximal coverage set*. The maximal coverage set is so-named because all signals of the set "cover" the point in question if the random quantities (signal amplitude and noise level) simultaneously satisfy the appropriate coverage access criterion. In the same sense, there is a finite probability that the actual coverage set could be empty if all random quantities simultaneously failed to satisfy the appropriate access criteria.

2.1.4 User Regional Priority Sub-model

The user regional priority sub-model introduces another type of spatial dependence in the system availability model. Through the use of relative weights, this sub-model specifies the probability of Omega usage in different geographical regions. The sub-model then combines the regionally-defined P_{SA} into an overall P_{SA} using the weights. Care must be exercised in interpreting combined probabilities governing separate spatial cells. For example, P_{SA} is not the probability that signal coverage simultaneously exists in all cells on the globe; it is also not the probability that coverage exists in at least one cell over the globe. P_{SA} is the probability that a user is in any particular cell on the globe at any time and that the particular cell exhibits coverage at that time. In practice, the time and space domains may be restricted, e.g., one or more hours in a month and/or a certain region of the globe, but this restriction is properly handled by normalization and does not affect the basic theory. Since P_{SA} is targeted to the individual user, the probability of his location in space and time is therefore important to the calculation. If a given Omega user has no particular predisposition for any cell/time, the probabilities are all equal (uniform weighting) and P_{SA} is proportional to the sum of the P_{SA} components computed at each cell and time. Most users do have geographical preferences/needs, however, so that the probability of utilizing Omega in a given cell varies from cell to cell. Preferences in time are much less common, although some user classes may exhibit more local daytime usage than local nighttime.

The probability of utilizing Omega in each of the 444 global cells is specified by a cell weighting matrix. For PACE applications, the weights associated with each cell are chosen as integral values between 0 and 10. To represent utilization probabilities, a selected set of weights is normalized over the globe. A region may be selected for evaluation of P_{SAA} by assigning non-zero weights to the appropriate cells and zero weights to all other cells. An example of cell weighting for a portion of the globe is shown in Fig. 2.1-1.

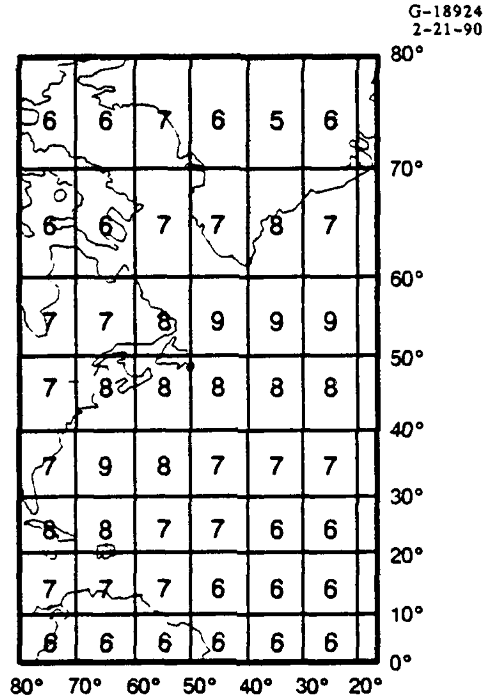

Figure 2.1-1 Example of Regional Weighting: Omega Civil Use

2.2 NEW SUB-MODELS FOR THE AUGMENTED MODEL

The sub-models reviewed in Section 2.1 are necessary in developing a system availability model which addresses signal accessibility only. However, additional sub-models are necessary for a system availability model incorporating position/navigation accuracy. Thus, calculation of P_{SAA} involves not only signal detection probability but also error in the predicted phase and the transformation of this phase error to position/navigation error. Sub-models governing phase error and position/navigation estimation are therefore addressed in the next two subsections.

2.2.1 Phase Error Sub-model

Since an Omega transmitting station radiates a single frequency signal at any given time, the only useful navigation information which can be passed to the user (assumed synchronized to the Omega pattern) is the signal phase. Most navigational filter algorithms assume a so-called "nominal" model of VLF phase variation with distance. In this model, the ratio of cumulative signal phase to distance from the transmitting station is fixed, i.e., independent of space, time, or direction. Since this nominal model does not describe real phase variation, a correction,

known as the propagation correction, or PPC, is applied to the phase measurement. The calculation of the PPC is based on a semi-empirical model of phase variation as a function of the electromagnetic characteristics of a signal path from transmitter to receiver.

Thus, by definition, the PPC and the nominal model together determine the predicted phase for a given station, signal frequency, position, and time. The relationship is simply given by

$$\text{PREDICTED PHASE} = \text{NOMINAL PHASE} - \text{PPC} \tag{2.2-1}$$

In this relation, the nominal phase is the "dominant" term in the sense that it contains the cumulative phase from the signal source, i.e., transmitting station. Measured in cycles of signal wavelength (e.g., 10.2 kHz), the nominal phase is 100-500 for typical paths whereas the PPC is usually between −3.00 and +3.00, with a resolution of 0.01 cycle (a unit referred to as a centicycle). The predicted phase has a typical diurnal variation of 1-2 cycles, amounting to about 0.2-2% of the nominal phase.

Since they are predicted quantities, the PPCs contain errors, known as prediction errors which are given by (using Eq. 2.2-1)

$$\begin{aligned}\text{PREDICTION ERROR} &= \text{OBSERVED PHASE} - \text{PREDICTED PHASE} \\ &= \text{OBSERVED PHASE} + \text{PPC} - \text{NOMINAL PHASE}\end{aligned} \tag{2.2-2}$$

Two difficulties arise in determining the observed phase. The first is that a highly stable frequency standard (e.g., a Cesium standard) must be used in connection with an Omega monitor receiver which is synchronized to UTC (as are the Omega stations). Such standards are available at only a few Omega monitor sites throughout the world. Monitor sites associated with each transmitting station may be conveniently used for this purpose, since they record a distant (remote) signal phase with respect to the local station phase. For these monitor sites, the local station is approximately one wavelength from the monitor, and thus, the cumulative phase is time-invariant (signal does not interact with the ionosphere) and can be precisely measured. The second difficulty is that the total cumulative phase cannot be measured with a single frequency phase comparison system alone. In principle, direct measurement of cumulative phase is possible with a time-of-arrival measurement of the leading edge of the 1-second Omega pulse envelope or phase measurement of two synchronized, transmitted tones, separated in frequency by about 100 Hz. Since these methods are impractical, the usual procedure is to assume the nominal phase is correct to the nearest cycle. Although this may truncate the true prediction error, it is generally assumed to be a correct procedure.

Prediction error is composed of a bias and random component. The bias error component is obtained by averaging the observed phase expressed by Eq. 2.2-2 for a given station signal/frequency/monitor/hour over a 2-4 week period. This averaging procedure is permissible because the PPCs are roughly constant over a 2-4 week time duration. The associated random component is computed as the standard deviation of the prediction error over the same period. For signals in the maximal coverage set, bias errors, both positive and negative, typically range from 0 to more than 20 centicycles (cecs) at 10.2 kHz; random error standard deviations (always positive) for the same signals typically range from 3-10 cecs. For the shorter wavelength at 13.6 kHz, spatial displacements of the ionosphere will affect a greater fraction of the signal's spatial cycle, thus leading to larger centicycle errors at 13.6 kHz than 10.2 kHz. Although its wavelength is intermediate in length, 11-1/3 kHz signals generally have larger prediction errors than the other two frequencies because the database from which the PPC model coefficients were computed was much smaller (Ref. 6). In general, prediction errors are larger for

- Mixed paths (paths containing both day and night segments)
- Nighttime paths
- Transequatorial paths
- Westerly paths
- Longer paths (but shorter of the two great-circle paths).

Before attempting to model prediction errors probabilistically, the sources of these errors must be investigated. In considering possible sources of error, noise is usually first proposed as the cause of random (zero-mean) error. The relatively long processing times and narrow bandwidths common to Omega receivers, however, greatly reduce the impact of external atmospheric noise.

For example, consider a very marginal accessibility condition for a signal received in the presence of noise: a signal-to-noise ratio (SNR) of -20 dB in a 100 Hz bandwidth using a typical receiver on board an aircraft whose dynamics limit the designed receiver time constant to about 100 seconds. Since the duty cycle (fraction of time on-air) for an Omega signal is about 10%, the effective integration time is approximately 10 seconds. Now, it can be shown that for a receiver with an exponential filter (generally true for phase lock loop-type receivers) having a time constant of $1/\alpha$, the noise equivalent bandwidth (Ref. 7) is $\alpha/4$. Thus, for a time constant of 10 seconds, the effective bandwidth is 0.025 Hz. Since the noise referred to in the definition of

SNR is noise power per unit bandwidth, the SNR increase realized in reducing the bandwidth from 100 Hz to 0.025 Hz is

$$10 \log_{10} (100/0.025) = 36.02 \text{ dB}$$

Hence, the effective SNR at the receiver output is $-20 + 36 = 16$ dB. Furthermore, using a phasor model of signal phase error due to noise (see Appendix A), the standard deviation of phase error for a SNR of 16 dB is 1.8 centicycles. Even at the lowest threshold usually considered for phase detection, a SNR of -30 dB in a 100 Hz bandwidth, the phase error for the above example is only 5.8 cecs. Since the great majority of usable signals have SNRs better than -20 dB (100 Hz bandwidth), the phase error due to random atmospheric noise is nearly always better than 2 cecs.

Except for the worst case noted above, the phase errors arising from SNR considerations are significantly smaller than the observed day-to-day variation of signal phase recorded at Omega signal monitors. Table 2.1-1 shows bias and random 10.2 kHz phase errors as a function of UT hour on the Norway — Liberia path (and reciprocal path) for the month of April 1988. The bias error is simply the average error of the (unflagged) daily measurements at a fixed hour over the month. The random error is computed as the standard deviation of the daily measurements at a fixed hour over the month. This table shows that, for most hours, the bias error substantially exceeds the random error in magnitude. The monthly average SNR (in dB; 100 Hz bandwidth) is also shown for each hour in Table 2.2-1. The daily SNR "measurement" (at a given hour) is actually a receiver-computed estimate, based on the phase lock loop variation and a calibration curve. The computed phase error associated with the SNR is displayed in the table for comparison with the day-to-day variation. This phase error is computed as the root-sum-square of those phase error values associated with each day's SNR reading. In all cases, the day-to-day random phase variation exceeds that associated with the SNR. Table 2.2-2 displays, for the La Reunion — Argentina reciprocal paths in April of 1988, the same type of phase error information as Table 2.2-1. For these paths, it is noted that bias error is frequently smaller than the corresponding random (day-to-day) phase error. The random day-to-day phase errors here are also larger than the SNR-associated random error.

Tables 2.2-1 and 2.2-2 illustrate the principal components of phase error and the time scales over which they are defined. The random phase error component associated with the SNR is defined over a receiver time constant (typically, 1-5 minutes). Thus, for this component, the phase error is due to VLF atmospheric noise (in a 100 Hz bandwidth) propagated to the receiver within a few minutes of the observation time. The fact that the day-to-day variation in

Table 2.2-1 Comparison of 10.2 kHz Norway—Liberia Reciprocal Path Phase Errors from Three Sources: Prediction (Bias) Error, Day-to-day Variation (Random), and Phase Measurement Noise (function of the SNR)

LIBERIA
APRIL 1988
NORWAY (minus) LIBERIA
10.2 kHz

TIME (hour)	BIAS (OBS-PRD) (cecs)	RANDOM (STD DEV) (cecs)	SNR (MEAN) (dB)	PHS ERR (NOISE) (cecs)
0	4	6.2	−11.9	0.85
1	7	3.2	−11.6	0.83
2	3	5.6	−10.8	0.75
3	4	4.6	−11.2	0.74
4	9	3.7	−12.6	0.87
5	22	3.4	−17.0	1.47
6	41	3.9	−20.6	2.53
7	35	3.9	−10.2	0.67
8	21	2.0	−6.6	0.41
9	19	1.2	−5.3	0.35
10	19	2.2	−4.1	0.30
11	18	1.6	−3.9	0.28
12	18	2.3	−4.6	0.31
13	18	1.4	−6.3	0.39
14	18	1.3	−7.4	0.45
15	19	1.9	−9.7	0.62
16	20	1.7	−11.1	0.73
17	23	2.7	−12.2	0.85
18	21	3.3	−13.2	0.97
19	12	3.3	−12.4	0.97
20	15	3.9	−15.5	1.18
21	13	3.7	−17.3	1.67
22	9	4.7	−16.7	3.41
23	8	2.9	−13.0	1.04

NORWAY
APRIL 1988
LIBERIA (minus) NORWAY
10.2 kHz

TIME (hour)	BIAS (OBS-PRD) (cecs)	RANDOM (STD DEV) (cecs)	SNR (MEAN) (dB)	PHS ERR (NOISE) (cecs)
0	2	2.9	8.9	0.07
1	3	3.0	8.5	0.09
2	0	2.5	9.5	0.07
3	1	2.5	9.3	0.07
4	5	2.8	7.5	0.56
5	13	3.6	4.7	2.64
6	34	3.5	2.3	0.46
7	28	3.6	7.2	0.08
8	9	1.8	9.7	0.06
9	6	1.4	10.0	0.12
10	5	1.8	10.7	0.14
11	4	1.9	10.6	0.06
12	4	1.9	10.0	0.06
13	4	1.7	8.3	0.10
14	5	1.5	8.5	0.08
15	5	1.9	7.7	0.58
16	8	2.0	8.6	0.08
17	11	2.4	8.1	0.09
18	12	2.5	7.4	0.09
19	5	3.4	9.3	0.08
20	8	2.5	12.1	0.05
21	6	2.7	10.8	0.06
22	5	2.7	9.5	0.07
23	5	2.5	9.8	0.06

Table 2.2-2 **Comparison of 10.2 kHz La Reunion—Argentina Reciprocal Path Phase Errors from Three Sources: Prediction (Bias) Error, Day-to-day Variation (Random), and Phase Measurement Noise (function of the SNR)**

REUNION
APRIL 1988
ARGENTINA (minus) REUNION
10.2 kHz

TIME (hour)	BIAS (OBS-PRD) (cecs)	RANDOM (STD DEV) (cecs)	SNR (MEAN) (dB)	PHS ERR (NOISE) (cecs)
0	0	2.9	6.9	0.09
1	0	3.6	7.1	0.09
2	1	3.5	7.6	0.09
3	7	3.6	6.2	0.10
4	6	3.5	4.3	0.13
5	5	3.5	2.8	0.14
6	2	3.2	2.8	0.14
7	2	4.4	1.2	0.17
8	2	5.0	0.2	0.19
9	5	5.2	−1.7	0.24
10	12	6.8	−5.7	0.37
11	8	9.3	−7.5	0.47
12	5	6.6	−6.7	0.41
13	5	5.0	−7.4	0.44
14	4	5.8	−9.3	0.58
15	4	8.3	−7.8	0.48
16	6	9.1	−4.2	0.31
17	3	8.4	−1.2	0.22
18	2	7.2	0.4	0.19
19	3	6.3	1.5	0.17
20	3	5.5	3.4	0.14
21	1	3.8	4.7	0.12
22	0	3.4	6.4	0.10
23	0	3.4	6.8	0.09

ARGENTINA
APRIL 1988
REUNION (minus) ARGENTINA
10.2 kHz

TIME (hour)	BIAS (OBS-PRD) (cecs)	RANDOM (STD DEV) (cecs)	SNR (MEAN) (dB)	PHS ERR (NOISE) (cecs)
0	4	4.1	−2.2	0.35
1	6	3.3	−1.6	0.33
2	6	3.0	−1.2	0.32
3	8	3.5	−3.5	0.44
4	5	4.5	−5.4	0.60
5	2	3.7	−6.6	0.67
6	−1	3.1	−7.5	2.75
7	0	4.4	−7.2	2.75
8	1	7.3	−7.7	0.71
9	1	6.6	−10.3	2.81
10	9	3.1	−12.5	3.00
11	19	4.7	−12.5	2.96
12	14	5.4	−9.8	1.46
13	13	3.4	−9.2	0.99
14	3	3.0	−6.2	0.47
15	−2	11.5	−8.6	1.22
16	7	13.9	−6.5	0.51
17	6	10.2	−3.7	0.37
18	3	10.4	−3.7	0.37
19	3	10.9	−4.0	0.38
20	3	8.6	−3.6	0.38
21	0	4.9	−2.3	0.31
22	3	5.2	−0.8	0.25
23	4	4.1	−2.1	0.29

phase is larger than the phase error implied by each day's measured SNR suggests that the day-to-day variation is not attributable to noise, even allowing for possible errors in the SNR calibration curve. Day-to-day variations in the signal phase are presumably due to day-to-day differences in the ionosphere which serves as an upper boundary for the signal propagation path. Although the Omega signal wavelengths are long (approximately 30 km), and thus insensitive to small changes in the ionosphere, the paths are also long (typically, 100 wavelengths), which means they are subject to several different sources of variation, e.g., latitude-dependent magnetosphere-ionosphere interactions. These sources are presumably the same as those which give rise to the signal amplitude variations studied in connection with the enhanced system availability model (Ref. 2). As discussed above, the signal amplitude variations are assumed to be log-normally distributed with mean given by the theoretical prediction (from the signal coverage database). Thus, the "errors" (actual value — prediction) which result are symmetrically distributed (with respect to a logarithmically defined signal amplitude) about zero mean. Moreover, since amplitude errors and phase errors are logarithmically related (Ref. 8), the day-to-day random phase error (associated with the signal amplitude error) is expected to be normally distributed with a standard deviation determined by measurements. The phase bias error results from *prediction* error, however, in contrast to the signal amplitude error model, for which the prediction error is assumed to be zero. The phase bias error is therefore independent of the random phase error since the prediction uncertainties are, for the most part, unrelated to the sources of day-to-day variation. As with the standard deviation of the random error component, the phase bias error is difficult to model and is best obtained from measurement. Thus, neglecting the relatively small phase error component associated with the SNR, the phase errors are taken to be normally distributed with mean given by the phase bias error (measured) and standard deviation given by the random day-to-day variation (measured). In terms of a probability density function, the phase errors are distributed according to

$$p_{\Delta\phi}(\delta\phi) = \frac{e^{-(\delta\phi-\overline{\delta\phi})^2/2\sigma_\phi^2}}{\sigma_\phi\sqrt{2\pi}} \qquad (2.2\text{-}3)$$

where $\overline{\delta\phi}$ is the bias error and σ_ϕ is the random error standard deviation.

Equation 2.2-3 describes the density function for a single station signal at a given location and at a given time (month or half-month at a given hour). In the absence of a general model specifying the spatial dependence of phase error, the density function is strictly valid only at monitor sites where $\overline{\delta\phi}$ and σ_ϕ are available from measurements. In the time domain,

measurement of $\overline{\delta\phi}$ and σ_ϕ for a given hour/month/monitor site during a year or set of years presumably establishes the density function for the next year for the same hour/month/monitor site. However, a recent study (Ref. 9) suggests that projection of the phase error statistics ($\overline{\delta\phi}$ and σ_ϕ) from year to year may have a large uncertainty.

Calculation of the probability density function for position accuracy (upon which P_{SAA} is based) requires the joint phase error probability density function, i.e., an extension of Eq. 2.2-3 for multiple station signals (for a fixed frequency) at a given monitor site. The joint probability density function is constructed from the individual probability density functions through a known or assumed interdependence of signal phase errors. Since, in this case, the time and space dependence of signal phase error sources is not adequately known, the interdependence of multiple signal phase errors is also not known. Clearly, however, paths which are nearly identical will exhibit nearly identical phase errors. As the paths become more differentiated, the interdependence weakens, leading to eventual path independence. At the station monitors, some paths from remote stations have similar bearings, but large portions of the paths will not overlap. Thus, to a good approximation, signal phase errors (at the monitor sites) on multiple signals may be considered independent and the joint phase error probability density function may be written (for the signals in the maximal coverage set)

$$P_{\Delta\phi_1,\Delta\phi_2,\Delta\phi_m}(\delta\phi_1, \delta\phi_2, \ldots, \delta\phi_m) = \frac{e^{-\frac{1}{2}\sum_{i=1}^{m}\left(\frac{\delta\phi_i-\overline{\delta\phi_i}}{\sigma_i}\right)^2}}{(2\pi)^{m/2}\sigma_1\sigma_2\ldots\sigma_m} \tag{2.2-4}$$

where $\delta\phi_i$ is the phase error for station signal i, $\overline{\delta\phi_i}$ is the bias error for signal i, σ_i is the standard deviation of the day-to-day variation of the phase of signal i, and m is the number of signals in the maximal coverage set. In practice, the phase error domain is limited to a few cycles (depending on the monitor receiver information output to the recorder).

2.2.2 Position Estimation Model

Because the principal navigation users of Omega are from the air transportation community, the position estimation model used for the system availability algorithm parallels (as far as possible) the position estimation techniques mechanized for aircraft receiver systems. Although aircraft Omega receiver mechanizations differ between manufacturers, a generic scheme is described which is considered common to a large class of aircraft receivers. Some background is presented to motivate this method. The generic scheme must be defined in order to show how phase errors are converted to position errors in the normal course of navigation.

Omega users and receivers have employed a number of schemes to convert phase measurements to two-dimensional position on the surface of the earth. Before the advent of microprocessor–based receivers, hyperbolic techniques (mostly for surface applications) were used in which phase differences provided by the receiver identified charted lines-of-position (LOPs) which are actually segments of hyperbolas defined on the surface of a sphere. The user's position is assumed known to within the resolution of a "lane" (distance interval corresponding to a full cycle of phase difference) and navigation is performed by tracking the changing position and noting any lane changes. This method was primarily used because it eliminated the need for an expensive, on-board frequency standard and because most of the early users (marine craft) moved sufficiently slowly so that multiple fixes occurred within a lane. Also, manual methods are considerably less precise than analytical/numerical methods in estimating position when more than two LOPs are present.

The great differences in air and marine vehicle motions lead to different navigation/positioning schemes for Omega receiver systems on the two kinds of platforms. One important difference is that the faster vehicle speed permits sensing the change in single-station phase using a relatively inexpensive precision crystal oscillator. Thus, in a relatively short time, spatially-separated measurements are made which can be treated as quasi-independent expressions for the phase "arrival time" when the propagation path is at an essentially fixed global time. For example, an aircraft traveling toward a station at 200 knots using a receiver with a 2-minute time constant will effectively make two independent measurements during each 16 nautical mile lane. Hence, three independent measurements are made in a period of 6 minutes, a time interval during which the ionosphere over a typical path changes very little. Moreover, the distance between measurement updates (approximately 7 n.m. for this example) is short compared to the curvature of the earth so that a "flat-earth" approximation can be used. This property also permits linearizing the range equations so that position updates can be rapidly and efficiently processed from the phase measurements.

Thus, navigation of the airborne receiver can proceed from an initial known position by processing phase change measurements from two or more stations using the linearized range equations. A minimum of two station phase measurements is required because two unknown quantities appear in the linearized range equations: position change along two orthogonal surface coordinates (e.g., north/east or latitude/longitude). Accurate phase change measurements are possible if the receiver's internal clock on-board the aircraft is sufficiently stable within

successive updates. A typical requirement is that the oscillator "drift" between successive updates be less than one microsecond (approximately one centicycle at 10.2 kHz). For a two-minute time constant receiver, this requirement translates into a stability of approximately 8 parts in 10^9, which is well within the capability of most modern precision crystal oscillators. In an operational airborne receiver, the initial position (e.g., coordinates of the point of departure) is known but, once enroute, relatively few precision updates (obtained by external means) are available. Thus, between precision updates, the position error may grow, but not monotonically, since the phase errors have a complex (non-systematic) space/time dependence on the signal paths. The clock/oscillator drift between precision updates is usually well-approximated as a linear function of time (drift rate constant) and thus systematically grows between precision updates. For these and other reasons, most receiver implementations include the clock drift offset as a state variable. In this case, a minimum of three station signals are required. These ideas are quantified in the following development.

The phase change, $\Delta\phi$, between updates on a given station signal with respect to a receiver's clock/oscillator is given

$$\Delta\phi = \frac{\partial\phi}{\partial\alpha}\,\Delta\alpha + \frac{\partial\phi}{\partial T}\,\Delta T \qquad (2.2\text{-}5)$$

where α is the signal path length over the great circle between the transmitting station and the receiver and ΔT is the time between updates. Since PPCs are added to the phase measurement to remove space and time dependence (transformation to the nominal model), the first partial derivative in Eq. 2.2-5 is given as

$$\frac{\partial\phi}{\partial\alpha} = k_o$$

where k_o is the frequency-dependent nominal wave number.* Since the clock drift between updates is assumed linear, the second term partial derivative in Eq. 2.2-5 is

$$\frac{\partial\phi}{\partial T} = \gamma$$

where γ is the constant clock drift rate. Thus, Eq. 2.2-5 becomes

$$\Delta\phi = k_o\,\Delta\alpha + \gamma\Delta T \qquad (2.2\text{-}6)$$

*For example, at 10.2 kHz, k_o is about 0.034 cycle/km.

Since $\Delta\phi$ is measured by the receiver and ΔT is just the elapsed time between updates, the unknowns in Eq. 2.2-6 are $\Delta\alpha$ and γ.

The path length (α (radians)) on a spherical earth is obtained as the scalar product of unit position vectors for the receiver ($\hat{r}_R$) and transmitter ($\hat{r}_T$) in a geocentric coordinate system, i.e.,

$$\hat{r}_R \cdot \hat{r}_T = \cos\alpha \tag{2.2-7}$$

If the small change in $\hat{r}_R$ between updates is denoted as $\Delta\hat{r}_R$, then, to first order in $\Delta\hat{r}_R$ and $\Delta\alpha$, Eq. 2.2-7 implies (recall $\hat{r}_T$ is fixed)

$$\Delta\hat{r}_R \cdot \hat{r}_T = -\sin\alpha\,\Delta\alpha \tag{2.2-8}$$

Expanding $\Delta\hat{r}_R$ in local north and east coordinates (in the earth's tangent plane at $\bar{r}_R$), i.e.,

$$\Delta\hat{r}_R = \Delta N\hat{n} + \Delta E\hat{e}$$

[$\hat{n}$, $\hat{e}$ are unit vectors along north and east, respectively. ΔN and ΔE are the distance changes between updates along north and east, respectively.]

and substituting in Eq. 2.2-8 can be shown to result in

$$\cos\beta\,\Delta N + \sin\beta\,\Delta E = -\Delta\alpha$$

where β is the local station bearing (with respect to geographic north). Substituting this form for the change in signal path length (between updates) into Eq. 2.2-6 yields

$$\Delta\phi = -k_o\,(\cos\beta\Delta N + \sin\beta\Delta E) + \gamma\Delta T \tag{2.2-9}$$

Equation 2.2-9 contains three unknowns (ΔN, ΔE, and γ) and thus three signal phase measurements are required. When more than three signals are available, the redundant data are used to provide increased position accuracy, since the errors on each signal path (to each monitor) are assumed independent. For more than one signal, relations similar to Eq. 2.2-9 may be written in matrix form, viz.

$$\Delta\phi = H\,\Delta X' + v$$

where: $\Delta\phi$ is a vector whose components, $\Delta\phi_i$, i = 1,2,...,8, are the changes in phase for station signal i (for a given frequency) between successive navigation updates, H is the measurement matrix whose components are given by

$$H_{i1} = k_o\cos\beta_i,\ H_{i2} = k_o\sin\beta_i,\ H_{i3} = \Delta T,\ i = 1, 2, \ldots 8,$$

β_i is the geographic bearing to the ith station, $\Delta X'$ is a 3-component vector in which $(\Delta X')_1 = \Delta N$, $(\Delta X')_2 = \Delta E$, and $(\Delta X')_3 = \gamma$, the minus sign is absorbed in k_o, and v is the zero-mean measurement noise vector.

In most receiver systems, position change, and clock drift are estimated from redundant phase data using a least squares method. For this technique, estimates of ΔN, ΔE, and γ are sought which minimize $E(v^T W v)$, where W is a symmetric weighting matrix which permits inter-channel measurement noise coupling and E( ) indicates expectation over the noise statistics. The resulting estimates are given in terms of the measurements by

$$\Delta \hat{X}' = (H^T W H)^{-1} H^T W \Delta\phi \qquad (2.2\text{-}10)$$

In conventional systems these position change and clock drift estimates are filtered in software (e.g., a Kalman filter) to minimize the possibility of large, sudden excursions in position and clock drift. Many systems also include algorithms to deselect signals which are expected to contain dominant long-path or modal components. Thus, a reasonable working assumption is that signals actually processed for navigation are those in the maximal coverage set.

The position change estimation sub-model required for the system availability calculation parallels the above description of Omega signal processing for a generic aircraft receiver system. In particular, the least-squares position change/clock drift estimate (Eq. 2.2-10) based on measured phase changes is used but other assumptions are necessary to make the probabilistic model tractable. The principal assumptions for the sub-model are summarized as follows:

(1) The position change/clock drift corresponding to the measured phase change is based on the least-squares estimate, Eq. 2.2-10

(2) The initial position is assumed correct so that errors incurred in the final position are due solely to phase change errors transformed through the position change estimate

(3) Initially, the receiver clock offset (with respect to the Omega epoch) is zero so that the receiver clock is precisely synchronized with Omega time at the beginning of the update cycle

(4) No filtering or weighting of position estimates is performed.

Assumptions (2) - (4) are necessary to make the model scenario-independent. In a typical flight scenario, the known airport position is entered at departure and successive position updates are computed based on Omega phase measurements and other information; occasionally,

precision updates are made enroute when, for example, the aircraft overflies a known, surveyed position. Clearly, errors in position grow as the flight departs its initial position, although the error growth is not necessarily monotonic and is nearly always bounded. In a similar way, errors in the estimated clock drift can accumulate following flight departure. Thus, the error at a given update point along the flight route depends to a degree on the flight history (e.g., departure time/location). To remove this scenario dependence, the initial position and clock synchronization, at any point in the flight, are assumed to be correct. Similarly, filtering and weighting are processes which depend on previous measurement/update states and thus, also have a flight history dependence. As a result, no filtering/weighting are assumed. Assumption (2) is clearly optimistic in the sense that no errors are assumed to be carried over from the previous update cycle. This assumption is partially compensated for by assumption (3) which specifies no filtering or weighting. This means that successive position estimates (which incur error over only one update cycle) tend to "jump" around the true position and noisy signals are weighted the same as strong signals.

The phase change measured by the receiver (and modified by the PPC) may be separated into two components, i.e.,

$$\Delta\phi = \Delta_0\phi + \delta\phi$$

where $\Delta_0\phi$ is the "true" phase change and $\delta\phi$ is the phase change error. The least-squares estimate expression, Eq. 2.2-10, is linear, so that insertion of the above expression (in vector form) for $\Delta\phi$ into Eq. 2.2-10 yields (no weighting implies W is an identity matrix)

$$\Delta\hat{X}' = (H^TWH)^{-1}H^TW\,\Delta_0\phi + (H^TWH)^{-1}H^TW\,\delta\phi$$

The first term on the right-hand side of the above expression is the true position change since no errors are involved and redundant LOPs pass through the single intersection point. Thus, the second term is the position change/clock drift error, which may be written as

$$\delta X' = (H^TWH)^{-1}H^TW\,\delta\phi \qquad (2.2\text{-}11)$$

This expression is used to transform the phase error probability density function to an intermediate probability density function over position error and clock drift error. This intermediate density function is integrated over clock drift error to obtain the position error probability density function (Appendix B).

2.3 POSITION ERROR PROBABILITY FUNCTIONS

In this section the position error probability density function obtained from the signal phase error model and position estimation model is briefly presented. Since the full derivation of the density function is given in Appendix B, only a descriptive account is included in this section. Conventional measures of position error which can be expressed in terms of the position error density function are compared and interpreted. Finally, P_{SAA} is expressed in terms of the position error distribution function.

2.3.1 Position Error Density Function

In Section 2.2.1, it is shown that the signal phase error at a given monitor site is described by a joint normal distribution over the independent station signal paths (Eq. 2.2-4). The signals contained in this joint distribution are those in the maximal coverage set. The position estimation model (Section 2.2.2) dictates how phase change errors for two or more signals (those in the maximal coverage set) are transformed to the two orthogonal components of position error and the single component of clock drift error. Since the transformation is linear and the phase errors are normally distributed, the position and clock drift errors are also normally distributed as described by an intermediate density function (see Appendix B). To obtain the position error density function, the intermediate density function is integrated over the clock drift error.

Thus, the position error density function is a two-dimensional normal distribution of the form

$$p_{\Delta X}(\delta X) = \frac{1}{2\pi} \sqrt{\det Q}\ e^{-\frac{1}{2}(\delta X - \overline{\delta X})^T Q (\delta X - \overline{\delta X})} \qquad (2.3\text{-}1)$$

where δX is the position error vector containing the north and east components, Q is a 2x2 matrix which describes how the position errors vary with direction (the determinant of Q indicates the "spread" of the distribution), and $\overline{\delta X}$ is the bias error vector with north and east components. The quantities Q and $\overline{\delta X}$ are functions of the phase error biases and standard deviations, and station bearing angles for all station signals in the maximal coverage set. The distance scale is determined by the frequency-dependent nominal wavenumber, k_o, normally expressed as centicycles/kilometer. The time scale is set by the time between updates.

Inspection of Eq. 2.3-1 shows that the position error density function is constant for those δN, δE for which

$$(\delta X - \overline{\delta X})^T Q (\delta X - \overline{\delta X}) = c \text{ (constant)} \qquad (2.3\text{-}2)$$

In terms of the two dimensions, δN and δE, Eq. 2.3-2 represents conic sections centered about the point $\overline{\delta E}$, $\overline{\delta N}$. If the eigenvalues of Q are all positive, Eq. 2.3-2 describes an ellipse. The ellipse is centered at ($\overline{\delta E}$, $\overline{\delta N}$) and, in general appears rotated with respect to the north and east error axes (see Fig. 2.3-1). As the constant c in Eq. 2.3-2 is increased (decreased) the size of the ellipse decreases (increases). The determinant or trace of Q inverse describes the relative "peakedness" of the distribution, e.g., if $\det Q^{-1}$ is small (large), then the distribution is relatively peaked (flat). If the eigenvalues of Q differ significantly in magnitude, the ellipse will be elongated in one direction with respect to the orthogonal direction. Appendix B contains additional information concerning this density function.

2.3.2 Conventional Error Measures

The position error density function provides a useful basis for the discussion and comparison of conventional error measures since virtually all error measures can be derived from this function. Different measures of error arise because particular applications evaluate alternatives/tests based on specific quantitative features of the position error density function. Several kinds of error measures commonly used in the analysis of navigation/positioning data are defined and discussed below.

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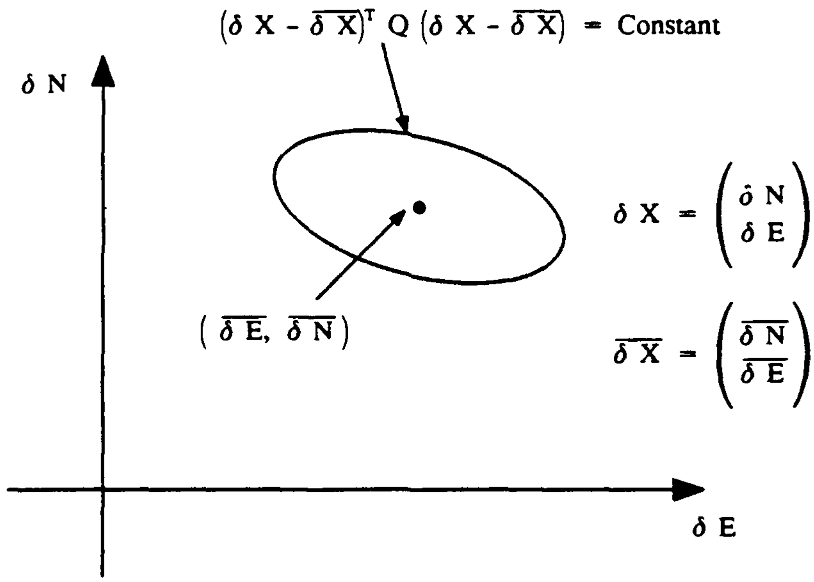

Figure 2.3-1 Cross Section of General Position Error Density Function in Plane Parallel to Earth's Tangent Plane at an Arbitrary Location

1. Fix bias error — the error obtained by averaging the coordinate errors (i.e., $\overline{\delta N}$ and $\overline{\delta E}$ in the density function, Eq. 2.3-1) for each position measurement and comparing with the true position (the origin on an error plot). For the density function the magnitude of the fix bias error is

$$\sqrt{(\overline{\delta N})^2 + (\overline{\delta E})^2}$$

In many systems, this component of the error is small, or assumed small, but in Omega applications, the fix bias error is substantial, frequently larger than the scatter about the mean (see measure (2) below). Note that this error measure is a vector, i.e., specified by two coordinates. The point defined by the coordinates $(\overline{\delta N}, \overline{\delta E})$ is known as the fix bias point.

[In the descriptions which follow, the error measures are referenced to two positions: (1) the true position (coordinate origin) and (2) the coordinates of the fix bias point.]

2. RMS error — a common error measure obtained as the square root of the mean square error magnitude over the position measurements. When referenced to the fix bias point, this measure is the same as the *standard deviation*, i.e., (using the above notation)

$$\sqrt{E\left((\delta X - \overline{\delta X})^T (\delta X - \overline{\delta X})\right)}$$

This quantity is a measure of scatter about the mean of the distribution and is sometimes referred to as the "random" component of position error. When referenced to the true position, this error measure is often called "dRMS." Multiples of this figure are frequently quoted, e.g., "the 2 dRMS error experienced by the receiver was ... ," especially when referring to accuracy criteria. Note that this error measure is a scalar (specified by one quantity).

3. Mean radial error — an error measure sometimes defined as the average "miss distance," i.e., the error magnitude, averaged over all position error measurements. This error measure is almost always referenced to the true position. In terms of the notation used above, the mean radial error is

$$E\left[\sqrt{\delta X^T \delta X}\right]$$

and, hence is larger than the fix bias error and smaller than the RMS (about the true position). This error measure is a scalar.

4. Most probable error — an error measure coinciding with the *mode* of the error distribution, i.e., that error corresponding to the coordinates of the peak in the position error density function. This is the error which would be most commonly observed among a large number of position measurements. Since the position errors are normally distributed, the maximum of the error density function occurs at the coordinates of the mean value, i.e., $\overline{\delta E}$, $\overline{\delta N}$. Thus, the most probable error corresponds to the fix bias error. This error measure is also a vector, but is sometimes referred to by its magnitude.

5. Circular error probable (CEP) — a two-dimensional error measure (originally applied to targeting accuracy) which is the radius of a circle enclosing a certain fraction (α) of the measurements. Historically, α was usually taken as 1/2 and the circle center coincided with the true position. In terms of the position error density function, the CEP is the radius of the cylinder which encloses volume α under the density function surface (total volume is one). A generalized CEP can be defined in which α can be any value between 0 and 1 and the cylinder centered about any point. Analytically, the generalized CEP is that value of r which satisfies

$$\alpha = \int_0^{2\pi} \int_0^{r} p(r', \theta) r' dr' d\theta \qquad (2.3\text{-}3)$$

where $p(r', \theta)$ is the polar form of the position error density function, Eq. 2.3-1. From this definition, it is seen that the CEP varies with the position of the cylinder center. In particular, the CEP is smallest when the cylinder center corresponds to the maximum of the position error density function, i.e., the fix bias point (or distribution mode). For $\alpha = 0.5$ and reference at the origin, the CEP corresponds to the *median* of the two-dimensional distribution in a radial sense, i.e., the cylinder divides the volume under the probability density function in half. The CEP is a scalar, but the specification of the generalized CEP requires 3 quantities, α and the two coordinates of the cylinder center.

A difficulty which arises in quoting and interpreting errors is the presumed relationship between the above error measures based on an assumed distribution of errors. For example, assuming a two-dimensional normal error density function having no bias error and circular cross section (eigenvalues of Q equal in Eq. 2.3-1), the probability that the radial error is within: (1) one standard deviation (or RMS; see error measure (2) above) of the true position is about 0.632; (2) two standard deviations of the true position is 0.982. However, these same numerical results are often applied even when the error distribution is non-circular and has a significant bias component.

As an example, consider a simple scenario (Fig. 2.3-2) in which the fix bias point is north of the true position, the error distribution is symmetric with respect to north and east, and the error density function has elliptical cross section. In this case, the matrix Q (see Eq. 2.3-1) is diagonal and has eigenvalues corresponding to the reciprocals of the north and east position error variances (assumed in this example to be one and two n.m., respectively). Figure 2.3-2 shows an elliptical contour resulting from a slice (parallel to the local earth tangent plane) through the position error density function at a probability density magnitude of $(1/4\pi)\exp(-0.5)$. Table 2.3-1

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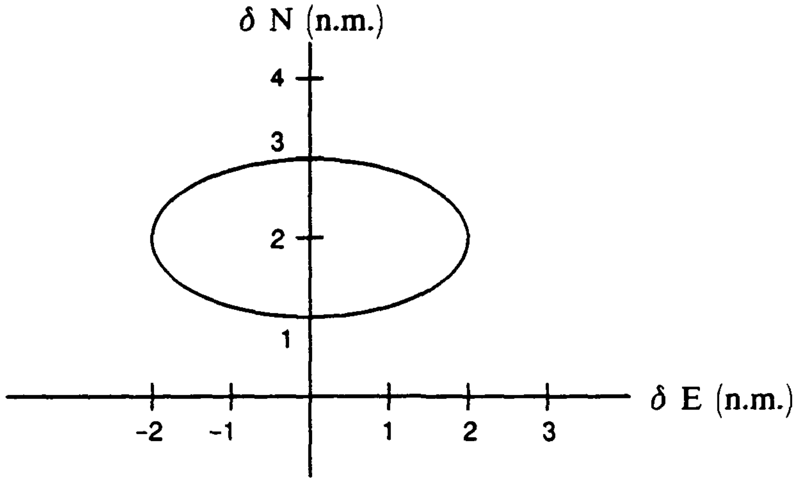

Figure 2.3-2 Contour of Sample Position Error Density Function Showing Bias Error and Unequal North and East Variances

Table 2.3-1 Probability that Radial Error is Within One and Two RMS Values of the True Position for Several Values of Bias Error and North/East Standard Deviation

NORTH BIAS ERROR (n.m.)	NORTH ERROR STANDARD DEVIATION (n.m.)	EAST ERROR STANDARD DEVIATION (n.m.)	RMS ERROR (n.m.)	RADIUS OF CIRCLE ABOUT TRUE POSITION (Units of RMS)	PROBABILITY THAT ERROR IS WITHIN CIRCLE
0	1	2	2.24	1	0.663
0	1	2	2.24	2	0.970
2	1	2	3	1	0.649
2	1	2	3	2	0.974
2	0.5	1	2.29	1	0.593
2	0.5	1	2.29	2	0.994
2	0.25	0.5	2.08	1	0.550
2	0.25	0.5	2.08	2	0.999

compares probabilities of errors within one and two RMS values of the true position for various bias errors and north/east standard deviations. For a non-zero north bias error ($\overline{\delta N}$), the RMS value referenced to the true position is given by

$$\text{RMS} = \sqrt{\sigma_E^2 + \sigma_N^2 + \overline{\delta N}^2}$$

The first two entries in the table show the probabilities of errors within one and two RMS values for no bias error but unequal standard deviations. The resulting probabilities differ by only a few percent from the circular, non-bias error case mentioned above. In the succeeding three pairs of entries the bias error is fixed at 2 n.m., but the standard deviations are successively halved. This corresponds to successively greater departures from the circular, non-bias error case, since the distribution becomes more peaked at a fixed offset from the true position. The table shows that the probability of error within one RMS value decreases as the distribution becomes sharper and the probability of error within two RMS values increases to almost one. Thus, improper characterization of the error distribution (e.g., by assuming no bias error and equal north and east random error standard deviations) can lead to erroneous numerical relations between the radius (in RMS units) of a circle about the true position and the probability of error within that circle.

2.3.3 Position Error Distribution Function and Definition of P_{SAA}

If the position error density function (Eq. 2.3-1) is converted to polar form, as in Eq. 2.3-3, and integrated over angle, the result is known as the radial error probability density function, i.e.,

$$p_R(r) = \int_0^{2\pi} p_{\Delta E,\Delta N} \, (\delta E(r,\theta), \; \delta N(r,\theta)) r \; d\theta$$

In this form, $p_R(r)$ dr gives the probability that the radial error is between r and r+dr. If this quantity is integrated from 0 to the radial error threshold, R_T, the radial error distribution function (frequently called the position error distribution function) is obtained, i.e.,

$$P(r \le R_T) = \int_0^{R_T} P_R(r) dr \qquad (2.3\text{-}4)$$

The position error distribution function is the essential ingredient in the definition of the augmented system availability index, P_{SAA}, since it sets a probabilistic condition on achieving navigation accuracy within pre-specified limits. If R_T is set too large, P_{SAA} values for most conditions would be very close to 1, i.e., 0.99999..., and, hence, not well differentiated. Similarly, if R_T is set too small, then the P_{SAA} values are also not well differentiated. More importantly, R_T may not represent a "real" error threshold because of the simplifying assumptions made in the position change estimation model. These assumptions are necessary to exclude any navigational history (scenario) dependence from the model but the actual error depends to a degree on the

navigational error. As a result, R_T should be selected, not in an absolute sense, but so as to permit comparison of P_{SAA} under various conditions.

To fully define P_{SAA}, the other probabilistic characteristics of the signals/signal reception, described by the remaining sub-models must be incorporated. First, define the event C as

C $\equiv$ event that, at any point in space and time, an Omega user experiences a radial position error less than a threshold value, R_T.

Clearly, event C depends on the signals available/used at the given point in space and time. The signals in the maximal coverage set, obtained from the coverage sub-model, serve as the pool of available signals to be processed by the receiver. Of these available signals, the signal coverage sub-model specifies probabilistic conditions under which the signal SNR is sufficiently high to permit phase tracking. The m signals in the maximal coverage set are labeled by the indices i_1, i_2, ... , i_m, where each index represents a different station signal (from the eight available. for a given frequency). Define the event U for a given signal subset of the maximal coverage set as

$U_{i_1 i_2 \ldots i_s} \equiv$ event that signals i_1, i_2, ..., i_s ($s \leq m$) are usable and all other signals in the maximal coverage set are not usable.

Since the s signals are a subset of the maximal coverage set, a usable signal in this context means that the corresponding station is on-air and the signal SNR is above the pre-specified threshold.

With the above definitions, the probability of experiencing an error less than R_T may be calculated, assuming that the receiver processes any available set of three or more signals (with SNR greater than threshold) in the maximal coverage set. Thus, it can be shown that

$$P(C) = \sum_{\{s\}} P(C\ U_{i_1 i_2 \ldots i_s}) \tag{2.3-5}$$

where the set {s} includes all combinations of three or more station signal indices within the maximal coverage set. By Bayes' theorem, each term in Eq. 2.3-5 may be expressed as

$$P(C\ U_{i_1 i_2 \ldots i_s}) = P(C \mid U_{i_1 i_2 \ldots i_s})\ P(U_{i_1 i_2 \ldots i_s}) \tag{2.3-6}$$

The first factor above on the right-hand side is just the position error distribution function, Eq. 2.3-4, computed for station signals i_1, i_2, ..., i_s. To illustrate how the second factor is computed, consider a simple example in which the maximal coverage set consists of signals

from stations 1, 2, 3, 4. Of the four three-station subsets in this example, consider first the event U_{123}. This event is expressed in terms of events defined in connection with the non-deterministic signal coverage sub-model and the station reliability/availability sub-model. For SNR considerations, the event A is defined so as to be consistent with the development in Ref. 2, i.e.,

$$A_i \equiv \text{event that } S_i/N > \text{THR} \;;\; \overline{A}_i = \text{event that } S_i/N < \text{THR}$$

where S_i is the signal level from station i, N is the noise level in a 100 Hz BW about the signal frequency (at the given time and location), and THR is the pre-specified lower-bound SNR threshold. From the station reliability sub-model, T-events are defined so as to be consistent with Ref. 1, i.e.,

$$T_i \equiv \text{event that station i is on-air } (i = 1,2,\ldots,8)$$

$$\overline{T}_i \equiv \text{event that station i is off-air } (i = 1,2,\ldots,8)$$

In terms of these events, the event U_{123} may be written (event product indicates intersection and sum denotes union)

$$U_{123} = A_1T_1A_2T_2A_3T_3\left(\overline{A}_4T_4 + \overline{T}_4\right)$$

Each of the first three pairs of factors in this expression indicates the event that the station is on-air *and* the SNR is greater than the threshold value. The factor in parentheses expresses the event that signal 4 is not available either because the SNR is below threshold (with the station on-air) *or* the station is off-air. Carrying out the indicated multiplication (set intersection) yields two events which are mutually exclusive. This simplifies the probability calculation, yielding

$$P(U_{123}) = P(A_1A_2A_3\overline{A}_4T_1T_2T_3T_4) + P(A_1A_2A_3T_1T_2T_3\overline{T}_4) \qquad (2.3\text{-}7)$$

The events $\{A_i\}$, $\{\overline{A}_i\}$ are caused by signal and noise propagation phenomena as well as receiver characteristics, independent of the operational causes for the station off-air events (T-events). Thus Eq. 2.3-7 becomes

$$P(U_{123}) = P(A_1A_2A_3\overline{A}_4)\; P(T_1T_2T_3T_4) + P(A_1A_2A_3)\; P(T_1T_2T_3\overline{T}_4) \qquad (2.3\text{-}8)$$

The first factors in the two terms on the right-hand side of this expression may be computed in terms of the mean noise level, noise standard deviation, mean signal amplitude, and signal amplitude standard deviation from the signal coverage sub-model, using analytical techniques described in Ref. 2. The corresponding second factors may be computed using data from the station reliability sub-model using algorithms indicated in Appendix A of Ref. 1. Expressions

similar to Eq. 2.3-8 may be obtained for the other three three-signal event combinations and the four-signal event combination, U_{1234}. These probabilities are inserted in Eq. 2.3-6, which, when combined with the appropriate position error distribution function value, is used in Eq. 2.3-5 to calculate P(C).

The above example traces the calculation of P(C) for a maximal coverage set of four stations. Calculations for larger maximal coverage sets proceed similarly, except that the number of terms to be computed grows rapidly. For example, there are two terms in Eq. 2.3-8 for one three-signal event. Counting all other signal combinations for the four-signal maximal coverage set gives a total of nine terms (see Appendix B (Section B.4)). For a maximal coverage set of five signals, the number of terms jumps to 51. The largest maximal coverage set of eight stations (extremely rare) yields 3489 such terms.

The causes of event C are assumed to be independent of the receiver reliability, since the receiver is modeled to cease functioning randomly (with given parameters), not to introduce phase errors. Thus, if P(C) is multiplied by the receiver reliability/availability, P_R, the result is P_{SAA} at a given location (cell) and time. The complete definition of P_{SAA} is obtained by invoking the user regional priority sub-model as explained in Section 2.1.4. This results in a weighting of P(C) for each cell, depending on the relative importance of that cell to the user for Omega navigation. The weighting, in general, depends on the particular user's needs/experience, but, in practice is common to large groups of users. The final result for P_{SAA} is then given as

$$P_{SAA} = \sum_{i=1}^{N} w_i \, P(C_i) \, P_R$$

where N is the number of cells in the region (which may be the globe), $\{w_i\}$ is the normalized cell weighting set, and C_i is the event that the position radial error is less than threshold R_T for cell i. The resulting P_{SAA} is for a fixed time, since, as noted in Section 2.1.4, user weightings over time are usually not meaningful. When used as an index for system performance evaluation, however, averages may be taken over hour/month as described in Ref. 2.

2.3.4 Summary of P_{SAA} Calculation

Because the calculation of P_{SAA} is complex and involves a number of sub-models, it is useful to review the sequence in which the sub-models are invoked and the assumptions required in their application. Figure 2.3-3 illustrates the steps involved in the P_{SAA} calculation. In its current state, the phase error sub-model supplies bias and random (day-to-day standard

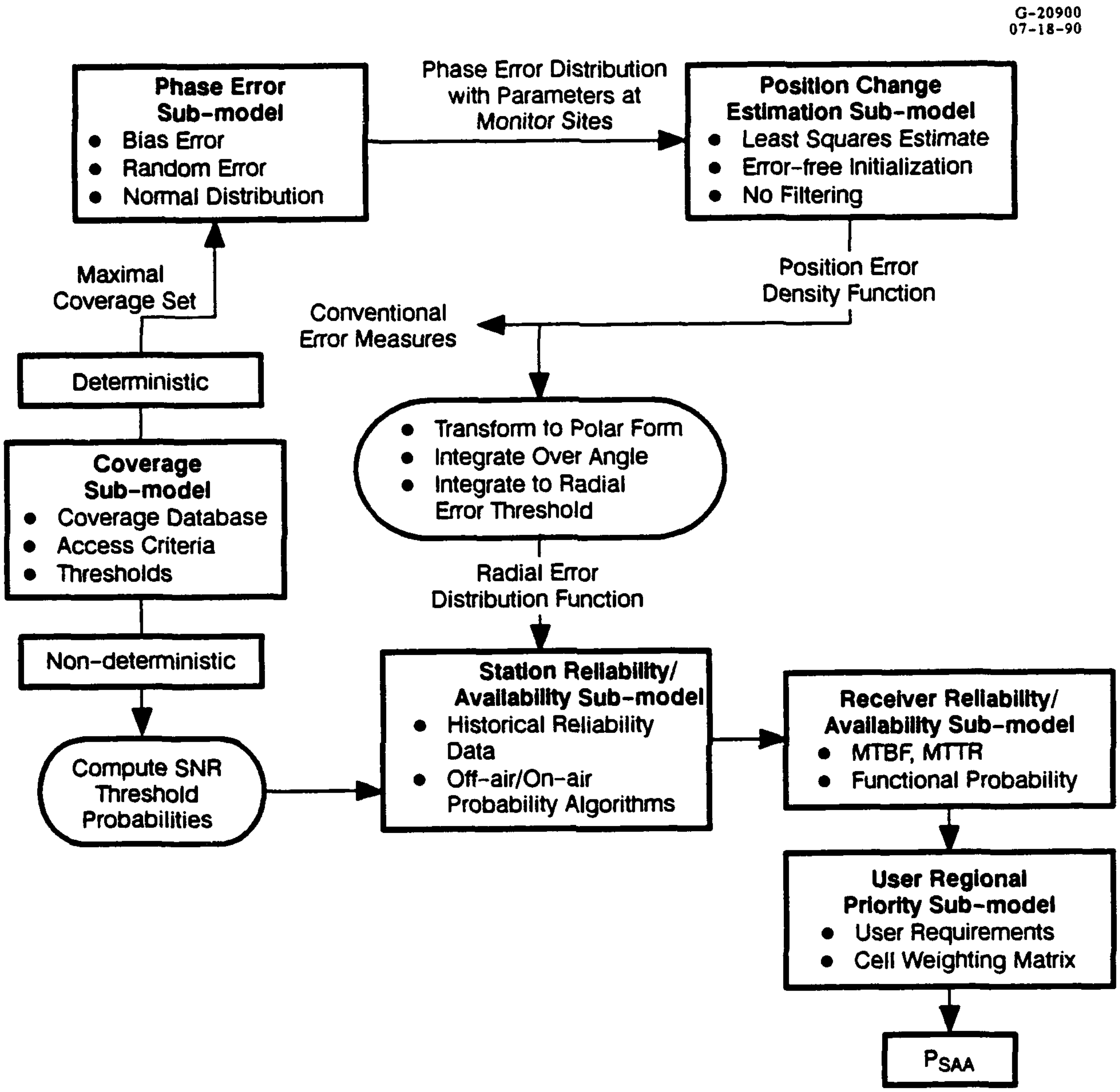

Figure 2.3-3 Procedure for Calculating P_{SAA} Using System Availability Sub-models

deviation) phase error data only at Omega monitor sites, thus severely limiting the spatial application of the model. Other sub-models, which may include a spatial dependence, are included, however, to completely define a structure for calculating P_{SAA} anywhere on the globe, whenever a satisfactory spatial model of phase error becomes available. The position change estimation sub-model describes how the phase error probability distribution is transformed to a position error distribution using a method reasonably similar to that employed in the navigation filters of conventional aircraft Omega receiver systems. The resulting position error density

function can be used to calculate conventional error measures (e.g., RMS error) or converted to a radial error distribution function. The coverage sub-model is used in two ways: (1) the deterministic portion of the sub-model establishes the maximal coverage set and (2) the non-deterministic portion is used to compute the probability that the SNRs associated with a given signal subset of the maximal coverage set exceed a designated threshold. Using the station reliability/availability sub-model, station on-air/off-air probabilities are combined with the the corresponding SNR threshold probabilities for the given signal subset to obtain the complete probability that the signal subset is available. The resulting probabilities are combined with the radial error distribution function (evaluated for the desired threshold error) for the appropriate signal subset and summed over all subsets of the maximal coverage set. If desired, the receiver reliability/availability sub-model can then be invoked to insert the receiver reliability factor which yields P_{SAA} for a given location and time. If phase error data is available on a global or regional basis (e.g., by use of a model fit to monitor data), then the user regional priority sub-model is used to obtain P_{SAA} for the globe or region.

2.4 COMPUTATIONAL RESOURCE ESTIMATES AND AN ALTERNATIVE ERROR MEASURE

Computational resources required for the calculation of P_{SAA} are expected to exceed those for the non-deterministic computation of P_{SA} (described in Ref. 2) primarily because additional sub-models involving additional calculations are required. Use of the phase error sub-model entails reading a small database to extract the appropriate bias and random error parameters. Transformation to the position error density function via the position change estimation model involves finding eigenvalues of a relatively small matrix, trigonometric calculations, and evaluating products of error functions. Integration of the position error density function over polar angle to obtain the radial error distribution function, however, must be carried out numerically for non-zero bias error. The computation of SNR threshold probabilities and station on-air probabilities is similar to that required for the non-deterministic P_{SA} computation. This stage of the calculation is the most time-consuming since numerical integrations must be performed for each signal subset of the maximal coverage set. Remaining calculations are minor.

From the above considerations, it can be seen that incorporation of the P_{SAA} calculation into an interactive workstation such as PACE depends on the feasibility of executing the non-deterministic P_{SA} calculation as a PACE option. Preliminary timing tests on the

non-deterministic P_{SA} computation, yield estimates of 0.7-2.0 minutes/cell on a PC/AT-type machine (assuming 5 stations in the maximal coverage set). More recent work suggests that these timing estimates may be optimistic by an order of magnitude. However, the deterministic signal approximation, described in Appendix C of Ref. 2, would reduce computation time by at least an order of magnitude if the approximation to the full random model (random signal, random noise) is found to be valid. Since the augmented system availability model, in its current state, is spatially restricted to the region in the immediate vicinity of each Omega monitor site, calculation of P_{SAA} is intrinsically limited to one or a few cells on the globe. As a consequence of the timing estimates and model limitations, the computation time for P_{SAA} is likely to be operationally feasible (a few minutes/cell) if the deterministic signal approximation is found to be valid.

If the deterministic signal approximation is judged to be invalid and P_{SAA} computation time becomes prohibitive, a simple calculation could easily be included into the PACE workstation which would provide some guidance on the relative accuracy afforded by different station configurations. The calculation would be derived from matrix Q first introduced in Eq. 2.3-1. As noted in Section 2.3, the determinant or trace of Q inverse furnishes information on the "spread" of the position error distribution. Specifically, the quantity

$$S = \frac{1}{\sqrt{\det Q}}$$

varies directly as the uncertainty of the position error density function; it is the product of the standard deviations of position error along each axis when Q is diagonal. To be applicable at locations other than monitor sites, the quantity must be independent of phase error data (random and bias component). This can be done by assuming the random component of phase errors equal for all signals and no bias error. In this case the constant phase error standard deviation (σ) and nominal wave number (k_o) can be absorbed into S to yield the dimensionless quantity

$$S' = \left(\frac{k_o}{\sigma}\right) S^{1/2}$$

The quantity S' is a function only of the bearing angles to each station whose corresponding signal is in the maximal coverage set. Defined in this way, S' is a measure of the geometric dilution of precision (GDOP). GDOP is a dimensionless measure of the ratio of linear position error to the uncertainty in the basic radionavigation signal resource, e.g., signal phase, time, or range. Thus, given a phase error σ, a measure of linear distance error may be obtained by multiplying

S' by σ/k_o. Note that S' becomes larger whenever one or more angles subtended by a pair of stations at the receiver becomes smaller. Although this quantity does not reflect phase bias error, unequal random signal phase errors, and other factors, S' does provide some indication of relative accuracy between different locations and different signal coverage sets. In a similar way, the quantity

$$V = \sqrt{\mathrm{Tr}\ Q^{-1}}$$

is a linear measure of error and, if the phase error standard deviations are assumed equal, the dimensionless form

$$V' = \frac{k_o}{\sigma} V$$

is obtained which depends only on the bearing angles to each station whose corresponding signal is in the maximal coverage set. V' is an error measure much like S', except that V' corresponds more closely to the conventional measure of GDOP than S'. In this sense, either measure could be used as a deterministic criterion for accepting signal subsets of the maximal coverage set. The procedure would be similar to applying the signal coverage criteria, except that the accuracy criterion would apply to multiple signals. The remaining part of the system availability calculation would proceed as described in Refs. 1 and 2.

The augmented system availability model has been developed in this chapter based on the four sub-models used in the original system availability model and two new sub-models dealing with phase error and position change estimation. Based on the current knowledge of the spatial dependence of phase errors, the system availability index for navigation accuracy, P_{SAA}, can only be computed at monitor sites where phase error measurements are available. In the development of an expression for P_{SAA}, the position error density function is derived and conventional error measures derivable from this quantity are described and compared. Estimates of the computational resources required for calculation of P_{SAA} are given and discussed. Simple, deterministic forms (similar to GDOP) are suggested as collective signal criteria should calculation of P_{SAA} become operationally prohibitive.

3. SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

3.1 SUMMARY

This report presents the development of an augmented system availability model for Omega which uses navigation accuracy as a system performance criterion/index. This criterion is believed to represent the most appropriate basis for system performance assessment since it focuses on the ultimate aim of navigation: accurate position tracking. The original (and enhanced) system availability model used signal coverage by three or more stations as the system performance criterion. Signal coverage by three or more stations, however, is a necessary but not a sufficient condition for navigation. Including navigation accuracy in the system availability model is not only appropriate from a user viewpoint but also is a common performance criterion for other radionavigation systems (see, e.g., Ref. 11).

To construct an augmented model of system availability, probabilistic sub-models of Omega navigation accuracy are required in addition to those needed for signal coverage. The signal phase error sub-model treats the major sources of phase error: PPC prediction (bias) error and the day-to-day phase variation at a fixed hour (random error). Other sources of phase error, such as the VLF atmospheric noise accompanying any given phase measurement, are found to be minor. The spatial dependence of the phase error is not known, however, so that random and bias error data are available only from certain monitor sites. The position change estimation sub-model uses a generic method for position tracking/estimation, similar in many respects to algorithms used in modern navigation filters, to transform the phase error statistics to a position error probability density function. As a digression, conventional error measures, such as RMS or CEP, are compared in terms of this density function. The position error density function is converted into a radial error probability distribution function which serves as a basis for P_{SAA} since it includes a threshold radial error. The probability that the radial error is less than the threshold value is computed assuming a given signal subset of the maximal coverage set is available. Thus, the probability distribution function for a given signal subset is multiplied by the joint probability that the SNR is above threshold for each signal in the subset, given that the corresponding station is on-air. A second multiplication is made by the joint probability that the stations corresponding to the signals in the maximal coverage set are on-air. Finally, P_{SAA} for a fixed location/cell is obtained by multiplying the result by the receiver reliability/availability

figure. Although the available phase error data restricts the spatial domain of the model, the user regional priority sub-model can be used in principle to define P_{SAA} for a region which may include the entire globe. P_{SAA} may also be integrated/averaged over time or evaluated at a single time (hour/month).

The computational resources required for the calculation of P_{SAA} are projected to be somewhat in excess of those required to compute P_{SA} for the non-deterministic case (Ref. 2). The calculation stage requiring the most computation time is expected to be the numerical integration associated with the SNR threshold calculation. Due to the spatial limitations of the model, P_{SAA} computation time is reduced compared to the global (444 cell) computations made in the deterministic model. For the machine hosting the PACE workstation, however, the processing time required for the full calculation of P_{SAA} is still expected to be too lengthy for operational use. Processing time can be further reduced by enlisting the deterministic signal approximation (Ref. 2), if it is determined to be valid. If the approximation is found to be invalid and the computation time is judged to be operationally unacceptable, simple alternative measures are proposed which are similar to the GDOP for a range-only system. Although these alternative measures are not probabilistic and do not consider phase bias error, variations in random error, and other factors, they provide an indicator of relative position accuracy among different receiver/multi-station configurations. If a threshold accuracy value is selected, the measures can be used as deterministic criteria for acceptability of a signal subset much like the access criteria in the signal coverage sub-model.

3.2 CONCLUSIONS

The development presented in this report demonstrates that an augmented system availability model can be formulated as a probabilistic model of Omega navigation accuracy. The development draws upon previously developed sub-models to establish probability of signal "coverage" and two new sub-models dealing with phase error statistics and the transformation from phase error to position error.

In its current state of development, the phase error sub-model furnishes statistics of phase bias and random error only at Omega monitor sites with stable (Cesium) frequency/clock references synchronized to UTC. Monitor sites with an unsynchronized Cesium clock reference could supply data on the random error component but not the bias error component. This restriction

limits the spatial application of the model to the region surrounding each synchronized monitor site over which the phase errors are highly correlated. This means that *neither a global model nor an arbitrary regional model of P_{SAA} is possible*, given the current state of knowledge of the spatial dependence of Omega phase errors. If such a spatial model of phase errors is developed in the future, however, it can be readily accommodated by the remaining model structure.

The position change estimation sub-model is found to be useful for this model and other error analyses since its features are similar to those of algorithms used in conventional navigational filters. In terms of applicability to modern systems, this sub-model represents an improvement over earlier analyses, which considered only fix errors incurred by processing phase differences (hyperbolic mode).

As in the enhanced system availability model, the most time-consuming stage of the performance index calculation is expected to be the calculation of the SNR threshold probabilities for each signal subset of the maximal coverage set, since this calculation involves a numerical integration over noise. A simplification of the non-deterministic coverage sub-model, known as the deterministic signal approximation (Appendix C of Ref. 2) could be employed to greatly reduce P_{SAA} computation time if the approximation is determined to be valid. Since P_{SAA} can only be computed in the immediate vicinity of the 8-10 synchronized monitor sites, based on our current understanding of the spatial dependence of signal phase errors, the calculation is reduced in scope (and, therefore, processing time) as compared to the full global calculations made by the deterministic system availability model.

Alternatives to the rigorous calculation of P_{SAA} presented in this report are simple GDOP-type expressions which are functions of the bearing angles to each station whose signal lies within the maximal coverage set. Although these simple forms ignore bias error and differences in random phase error on different paths, they have the advantage of permitting global or arbitrary regional calculation. With an appropriate threshold "GDOP" selected, these forms could be used as criteria to accept or reject signal subsets of the maximal coverage set in the calculation of deterministic/non-deterministic P_{SA}.

3.3 RECOMMENDATIONS

Based on the results and findings of this work, it is recommended that an algorithm for the calculation of a composite P_{SAA} be implemented (perhaps as an off-line feature of PACE)

by computing and averaging (unweighted/weighted) P_{SAA} values at the 10 synchronized monitor sites. Since these monitor sites (which are co-located with the Omega stations, in addition to New Zealand and South Africa) are reasonably well distributed throughout the globe, the resulting P_{SAA} would provide a useful sampling of the global P_{SAA} to serve as a monthly index of system performance. Because of its spatial limitation, however, it could not be used to evaluate the impact of options (such as station off-air/power reductions) on users in specific regions (e.g., the North Atlantic). The calculation of P_{SAA} is believed to be impractical, even as an off-line PACE option, without use of the deterministic signal approximation. It is therefore recommended that the validity of this approximation to the full calculation of SNR threshold probabilities (involving numerical integration) be explored and tested.

In addition to the above, it is recommended that either or both of the simple GDOP-type expressions be implemented as a PACE option. Because of their simplicity and expected ease of integration of these forms into the PACE structure, negligible additional processing time is anticipated for PACE operation. This option could be used with either the deterministic or non-deterministic P_{SA} computation options in PACE.

APPENDIX A

PHASE ERROR DUE TO RANDOM NOISE

Consider a system in which a very narrow bandwidth CW signal of amplitude S is detected in the presence of noise. The vector sum of the interfering noise fields (in a given narrow bandwidth) can be represented as a phasor with magnitude N and angle ϕ (see Fig. A-1). Without loss of generality, the signal phase can be taken to be zero. The received signal, which is the vector sum of the signal and the noise has magnitude

$$\sqrt{S^2 + N^2 + 2SN\cos\phi}$$

and phase angle ψ.

The signal-to-noise ratio (x, in dB) is expressed in terms of S and N by

$$x = 20\log_{10}(S/N) = \alpha\log_e(S/N) \qquad \text{(A-1)}$$

where $\alpha = 20\log_{10}e = 8.68589\ldots$. From Fig. A-1, the altitude, A, of the indicated triangles may be alternately expressed and equated to yield

$$N\sin\phi = (S + N\cos\phi)\tan\psi$$

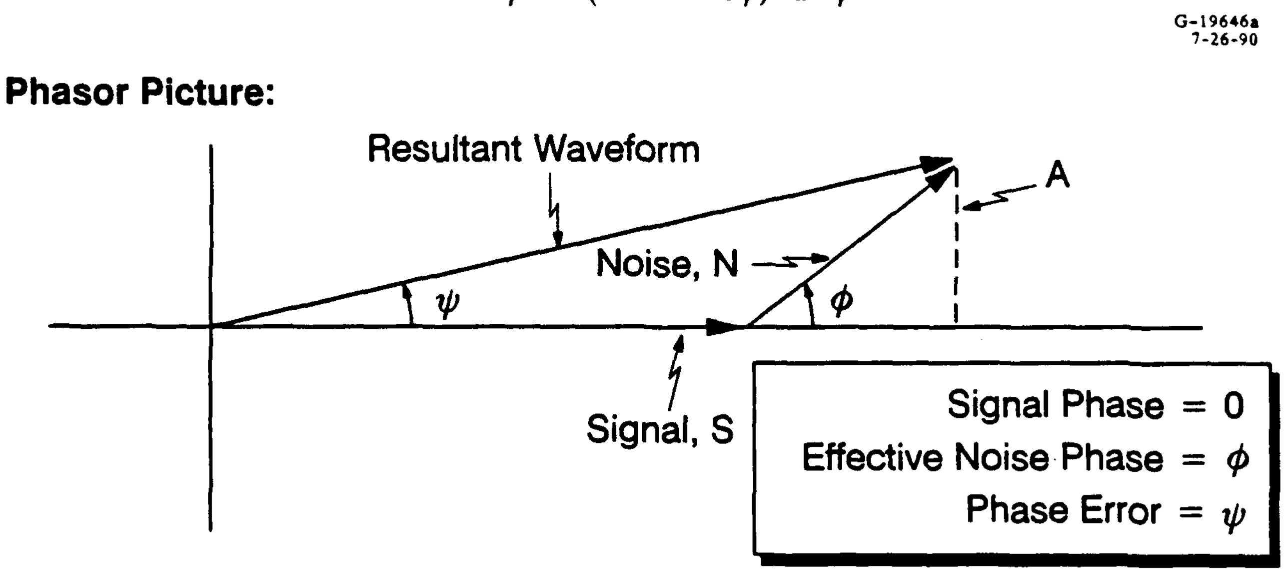

Figure A-1 Phasor Picture of Phase Error due to Noise

or

$$\tan\psi = \frac{\sin\phi}{(S/N) + \cos\phi}$$

Using Eq. A-1 for (S/N), the following expression is obtained for the phase error:

$$\psi = \arctan\left(\frac{\sin\phi}{e^{x/\alpha} + \cos\phi}\right)$$

If the noise phase angle is assumed to be random, i.e., described by a probability density function

$$p_\Phi(\phi) = \frac{1}{2\pi} \qquad 0 \le \phi \le 2\pi$$

then the first moment of ψ (i.e., the mean) may be expressed as

$$< \psi > = \int_0^{2\pi} \psi(\phi)\ p_\Phi(\phi)\ d\phi = 0$$

which follows from symmetry considerations or the fact that ψ is an odd function of ϕ. Since the mean vanishes, the second moment (variance) may be expressed as

$$< \psi^2 > = \int_0^{2\pi} \psi^2(\phi)\ p(\phi)\ d\phi = \frac{1}{2\pi}\int_0^{2\pi}\left[\arctan\left(\frac{\sin\phi}{e^{x/\alpha} + \cos\phi}\right)\right]^2 d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\arctan\left(\frac{\sin\phi}{e^{x/\alpha} + \cos\phi}\right)\right]^2 d\phi \qquad \text{(A-2)}$$

where the last equality follows because the integral is symmetric under the transformation $\phi \rightarrow \phi + \pi$. In terms of the variance, the standard deviation is given as

$$\sigma_\psi = \sqrt{< \psi^2 >}$$

The integral in the definition of $< \psi^2 >$, Eq. A-2, cannot, in general, be expressed in closed form so that numerical integration is required. For large SNR (given in dB by the quantity x), a small angle approximation to the arctangent may be used to considerably simplify the integration.

APPENDIX B

ANALYTICAL STRUCTURE OF THE AUGMENTED SYSTEM AVAILABILITY MODEL

In this appendix, the sub-models making up the augmented system availability model are used to develop the probabilistic form for the augmented system availability index, P_{SAA}. In its most general form this index gives the probability that an Omega user will experience an accuracy greater than a selected value at any location on the earth's surface at any time. For most applications, the time is either fixed or specified for some combination of hours and months. In this development, the time will be assumed fixed.

P_{SAA} is defined in terms of event D which is a function of the threshold radial error, R_T. $D(R_T)$ is defined as follows:

$D(R_T) \equiv$ event that a user navigating with Omega anywhere on the globe experiences a radial position error not greater than R_T.

B.1 USER REGIONAL PRIORITY SUB-MODEL

The geographic preferences of the Omega user are brought in to the development by first defining the spatial "universe" as the union of all events that a user is located in any two-dimensional cell on the globe. Here, a "cell" is defined as that area (approximately 600 n.m. × 600 n.m.) over which Omega signal accessibility (the ability to use any given set of Omega signals) changes very little. Thus, the spatial universe is expressed as:*

$$U^s = L_1 + L_2 + \dots + L_{444}$$

where L_i is the event that a user is located in cell i (444 such cells cover the globe). By definition of the set universe, the probability of event $D(R_T)$ is given as

$$P(D(R_T)) = P(D(R_T)U^s) = P\left(D(R_T)\sum_{i=1}^{444} L_i\right) = P\left(\sum_{i=1}^{444} D(R_T)L_i\right)$$

*In the following development with sets/events, sum indicates set union and product indicates set intersection.

Now, the events $D(R_T)L_1$, $D(R_T)L_2, \ldots$ are mutually exclusive since the single user being considered can only be located in one of the cells. Thus,

$$P(D(R_T)) = \sum_{i=1}^{444} P(D(R_T)L_i)$$

Using Bayes' Law, this expression may be written in terms of the conditional probability as

$$P(D(R_T)) = \sum_{i=1}^{444} P(D(R_T)|L_i)\ P(L_i) = \sum_{i=1}^{444} P(D_i(R_T))P(L_i)$$

where $D_i(R_T)$ is the event that an Omega user experiences a radial position error less than or equal to R_T given that the user is located in cell i. $P(L_i)$, the probability that the user is located in cell i, is usually written as w_i, where $\{w_i\}$ is a normalized set of weights indicating Omega user preference or experience.

Thus,

$$P(D(R_T)) = \sum_{i=1}^{444} w_i\ P(D_i(R_T)) \qquad ; \sum_{i=1}^{444} w_i = 1 \qquad \text{(B.1-1)}$$

B.2 RECEIVER RELIABILITY/AVAILABILITY SUB-MODEL

To include the receiver reliability/availability model, two additional events ($G_i(R_T)$ and F) are defined by the expression

$$D_i(R_T) = G_i(R_T)\ F$$

where $G_i(R_T)$ is the event that the Omega signals accessible to cell i can be processed to achieve a radial position error less than R_T and F is the event that the user's Omega receiver functions properly at the given fixed time. As before, the probability of event $D_i(R_T)$ may be written

$$P(D_i(R_T)) = P(G_i(R_T)|F)\ P(F)$$

To determine P(F), assume that all users are grouped into n_c receiver classes and define E_j as the event that the given user is in receiver class j. A receiver "class" is that group of receivers which have approximately the same reliability and detection sensitivity characteristics. As above, the universe of events E is the sum (union) $E_1 + E_2 + \ldots + E_{n_c}$ and thus F may be written

$$F = F(E_1 + E_2 + \ldots + E_{n_c}) = FE_1 + FE_2 + \ldots + FE_{n_c}$$

The events FE_1, FE_2, ... are all mutually exclusive since the single user being considered can only be in one class. Thus

$$P(F) = P\left(\sum_{j=1}^{n_c} FE_j\right) = \sum_{j=1}^{n_c} P(FE_j) = \sum_{j=1}^{n_c} P(F|E_j)\, P(E_j)$$

The probability, $P(E_j)$, that the user has a receiver in receiver class j is

$$P(E_j) = \frac{n_j}{N_u}$$

where n_j is the number of receivers in receiver class j and N_u is the total number of users in all receiver classes, i.e.,

$$N_u = \sum_{j=1}^{n_c} n_j$$

The conditional probability $P(F/E_j)$ is usually referred to as the receiver reliability figure for receiver class j and is often written P_{R_j}. Using a uniform failure interval and repair time model, it can be shown that the reliability figure for a receiver in receiver class j is

$$P_{R_j} = \frac{MTTR_j}{MTBF_j}$$

where $MTTR_j$ is the mean time to repair figure for receiver class j and $MTBF_j$ is the mean time between failures figure for receiver class j.

Thus,

$$P(F) = \frac{1}{N_u} \sum_{j=1}^{n_c} n_j\, P_{R_j}$$

and

$$P(D_i(R_T)) = \frac{P(G_i(R_T)|F)}{N_u} \sum_{j=1}^{n_c} n_j\, P_{R_j} \qquad \text{(B.2-1)}$$

B.3 SIGNAL COVERAGE SUB-MODEL

The signal coverage sub-model is next invoked by first defining

$$C_i(R_T) \equiv G_i(R_T)|F$$

where $C_i(R_T)$ is the event that the Omega user experiences a radial error of less than or equal to R_T in cell i given that the receiver functions properly. The signals that the user's receiver might possibly use in navigating with Omega in cell i at the given fixed time make up what is known as the *maximal coverage set*. The actual signals in this set depend on the thresholds selected for the access criteria applied to the coverage data. Typical access criteria are given as follows:

- Phase deviation (angle of phasor difference between Mode 1 signal and total signal) must be less than 20 centicycles
- Dominant mode must be Mode 1
- Ratio of total long-path signal amplitude to total short-path signal amplitude must be less than -3 dB
- Angle between the (great-circle) propagation path and the (great-circle) terminator must be greater than 5°.

All signals in the maximal coverage set, i.e., those which satisfy the above access criteria at a given cell and time (hour/month) are not necessarily usable by conventional receivers because of the random variations in the signal amplitude and noise level. A signal is said to be usable if, in addition to the above criteria, the following non-deterministic criterion is satisfied:

- Ratio of total short-path signal amplitude (assumed mean value) to median noise level (assumed mean value) in a 100 Hz BW must be greater than a certain threshold (typically, -20 dB).

The maximal coverage set is so-named because if all signals in the maximal coverage set happen to satisfy the non-deterministic criterion, then the usable subset of signals has its "maximal" value. On the other hand, it is possible (but not likely) that all signals in the maximal coverage set could simultaneously fail the non-deterministic criteria, meaning that no signals would be usable.

Of the signals in the maximal coverage set, the universe, U^v, of usable signal combinations are those subsets which contain at least three signals, i.e.,

$$\begin{aligned} U^v &= V_{i_1 i_2 i_3} + V_{i_1 i_2 i_4} + \ldots + V_{i_{m-2} i_{m-1} i_m} \\ &+ V_{i_1 i_2 i_3 i_4} + V_{i_1 i_2 i_3 i_5} + \ldots + V_{i_{m-3} i_{m-2} i_{m-1} i_m} \\ &+ \ldots. \\ &+ V_{i_1 i_2 i_3 \ldots i_m} \end{aligned}$$

where $V_{i_1 i_2 \ldots i_q}$ is the event that signals labeled $i_1, i_2, \ldots i_q$ are usable in cell i at the given fixed time. From their definition, the component events making up U^v are mutually exclusive. Thus

$$P(C_i(R_T)) = P(C_i(R_T)U^v) = P\left(C_i(R_T)V_{i_1i_2i_3} + C_i(R_T)V_{i_1i_2i_4} + \ldots\right)$$
$$= P\left(C_i(R_T)V_{i_1i_2i_3}\right) + P\left(C_i(R_T)V_{i_1i_2i_4}\right) + \ldots$$

In terms of conditional probabilities, the above expression becomes

$$P(C_i(R_T)) = P\left(C_i(R_T)\,|\,V_{i_1i_2i_3}\right) P\left(V_{i_1i_2i_3}\right) + P\left(C_i(R_T)\,|\,V_{i_1i_2i_4}\right) P(V_{i_1i_2i_4}) + \ldots \qquad \text{(B.3-1)}$$

The event $C_i(R_T)\,|\,V_{i_1i_2\ldots i_q}$ describes the situation in which an Omega user in cell i navigates with a radial error less than or equal to R_T using Omega signals $i_1, i_2, \ldots i_q$. Calculation of the associated event probability requires both the phase error and position change estimate sub-models which will be deferred until Sections B.5 and B.6. The probability of the V-events is calculated through use of the station reliability/availability sub-model as well as the signal coverage sub-model.

B.4 STATION RELIABILITY/AVAILABILITY SUB-MODEL

To show how the calculation of $P(V_{i_1i_2\ldots i_q})$ proceeds, it is instructive to consider two simple examples before addressing the general case. The first example is the case in which the maximal coverage set contains only 3 station signals. The second example considers a maximal coverage set with 4 station signals. Both the station reliability/availability and signal coverage sub-models are required for the calculations in these examples.

First, define the events A_i and $\overline{A}_i$ as follows:

$A_i \equiv$ event that the SNR for signal i exceeds the threshold, a

$\overline{A}_i \equiv$ event that the SNR for signal i is less than a.

The A-events are governed by the random fluctuations of signal and noise levels and do not account for the uncertainty in generating the signal itself. The events which describe the on-air/off-air status of the Omega stations are defined as follows:

$T_i \equiv$ event that station i is on-air

$T_i \equiv$ event that station i is off-air.

A maximal coverage set consisting of three station signals obviously contains only one subset of three or more station signals (a minimum of three station signals is assumed necessary

for Omega navigation). For convenience, the signals are labeled 1, 2, 3 (not necessarily corresponding to the usual Omega station number/letter convention). The event that the three signals are usable is given by

$$V_{123} \equiv (A_1T_1)(A_2T_2)(A_3T_3)$$

The events in parentheses indicate that the station is on-air *and* the signal SNR is above threshold. The probability of event V_{123} is written

$$P(V_{123}) = P(A_1A_2A_3T_1T_2T_3) = P(A_1A_2A_3)P(T_1T_2T_3) \ , \qquad \text{(B.4-1)}$$

since the A-events and T-events are independent.

The events A_1, A_2, A_3, however, are *not* independent since the noise level is common to the SNR associated with the three signals in the Omega receiver. For lognormally distributed signal amplitude and noise level, the result (derived in Ref. 2) is

$$P(A_1A_2A_3) = \int_{-\infty}^{+\infty} F_1(n)\ F_2(n)\ F_3(n)\ \frac{e^{-(n-\bar{n})^2/2\sigma_N{}^2}}{\sigma_N\sqrt{2\pi}}\ dn \qquad \text{(B.4-2)}$$

where

$$F_i(n) = (1/2)\ \text{erfc}\left(\frac{a-(\bar{s}_i-n)}{\sigma_{S_i}\sqrt{2}}\right) \quad ,$$

$\bar{s}_i$ is the mean signal amplitude (from the coverage database) for signal i, σ_{S_i} is the standard deviation of signal amplitude for signal i (from a special algorithm; see Ref. 2), $\bar{n}$ is the mean signal noise level and σ_N the noise level standard deviation (both included with the coverage database), and a is the SNR threshold. The function labeled "erfc" is the complimentary error function. These space/time-dependent parameters are specified for cell i at the given time.

The events T_1, T_2, T_3 are also not independent but for a more subtle reason. The off-air event $\bar{T}$ may be separated into mutually exclusive components, viz.,

$$\bar{T}_i = \bar{T}_i^u + \bar{T}_i^s$$

where:

$\bar{T}_i^u \equiv$ event that station i is in an *unscheduled* off-air status

$\bar{T}_i^s \equiv$ event that station i is in a *scheduled* off-air status.

An unscheduled off-air event at a given station is independent of unscheduled or scheduled off-air events at other stations. However, Omega operational doctrine bars the simultaneous occurrence of scheduled off-airs at different stations, i.e.,

$$\overline{T}_i^s \overline{T}_j^s = 0 \quad ; \; i, j = 1, 2, \ldots 8 \quad ; \; i \neq j$$

This last relation leads to a dependence between events T_i and T_j. It is shown in Ref. 1 that

$$P(T_1 T_2 T_3) = P(T_1 T_2)\left(1 - P(\overline{T}_3^u)\right) - P(\overline{T}_3^s)\left(1 - P(\overline{T}_1^u)\right)\left(1 - P(\overline{T}_2^u)\right) \tag{B.4-3}$$

where:

$$P(T_1 T_2) = P(T_1)P(T_2) - P(\overline{T}_1^s)P(\overline{T}_2^s)$$

which also shows explicitly the non-independence of T_1 and T_2. The individual off-air event probabilities $P(\overline{T}_i^u)$, $P(\overline{T}_i^s)$ are obtained from historical station reliability figures as explained in Ref. 1. Thus, Eqs. B.4-2 and B.4-3 are used in Eq. B.4-1 to compute $P(V_{123})$.

The second example considers a maximal coverage set of four station signals which contains four three-signal subsets and one four-signal subset. If the stations are labeled 1, 2, 3, 4 (similar to the previous example), then the first event subset may be written

$$V_1 = (A_1 T_1)(A_2 T_2)(A_3 T_3)\left(\overline{A}_4 T_4 + \overline{T}_4\right)$$

where the last event in parentheses means that signal 4 is not usable because either the SNR for that signal is below threshold (with the station on-air) *or* the station is off-air. The above expression for V_{123} may be written as the union of two mutually exclusive events:*

$$V_{123} = (A_1 A_2 A_3 \overline{A}_4)(T_1 T_2 T_3 T_4) + (A_1 A_2 A_3)(T_1 T_2 T_3 \overline{T}_4)$$

Since the two events are mutually exclusive, the probability of event V_{123} is

$$P(V_{123}) = P\left[\, (A_1 A_2 A_3 \overline{A}_4)(T_1 T_2 T_3 T_4) \,\right] + P\left[\, (A_1 A_2 A_3)(T_1 T_2 T_3 \overline{T}_4) \,\right]$$

and since the A-events and T-events are independent,

$$P(V_{123}) = P(A_1 A_2 A_3 \overline{A}_4)P(T_1 T_2 T_3 T_4) + P(A_1 A_2 A_3)P(T_1 T_2 T_3 \overline{T}_4) \tag{B.4-4}$$

---

*The events are mutually exclusive because T_4 intersects one event and $\overline{T}_4$ the other event.

To calculate $P(A_1A_2A_3\overline{A}_4)$, note that the set universe (for A-events) is $A_4 + \overline{A}_4$. Thus,

$$\begin{aligned} P(A_1A_2A_3) &= P(A_1A_2A_3(A_4 + \overline{A}_4)) \\ &= P(A_1A_2A_3A_4) + P(A_1A_2A_3\overline{A}_4) \end{aligned}$$

Hence,

$$P(A_1A_2A_3\overline{A}_4) = P(A_1A_2A_3) - P(A_1A_2A_3A_4) \tag{B.4-5}$$

where $P(A_1A_2A_3A_4)$ is given by

$$P(A_1A_2A_3A_4) = \int_{-\infty}^{+\infty} F_1(n)\ F_2(n)\ F_3(n)\ F_4(n)\ \frac{e^{-(n-\overline{n})^2/2\sigma_N{}^2}}{\sigma_N\sqrt{2\pi}}\ dn \tag{B.4-6}$$

and all quantities were defined in connection with Eq. B.4-2. Similarly, $P(T_1T_2T_3\overline{T}_4)$ can be written as

$$P(T_1T_2T_3\overline{T}_4) = P(T_1T_2T_3) - P(T_1T_2T_3T_4) \tag{B.4-7}$$

where $P(T_1T_2T_3T_4)$ is computed from $P(T_1T_2T_3)$ using the recursion formula (Ref. 1)

$$P(T_1T_2T_3T_4) = P(T_1T_2T_3)\left(1 - P(\overline{T}_4^u)\right) - P(\overline{T}_4^s)\left(1 - P(\overline{T}_1^u)\right)\left(1 - P(\overline{T}_2^u)\right)\left(1 - P(\overline{T}_3^u)\right) \tag{B.4-8}$$

With the use of Eqs. B.4-5, B.4-8, B.4-2, B.4-7, and B.4-3, $P(V_{123})$ may be calculated by means of Eq. B.4-4. In the same way the probabilities of the other usable 3-signal subset events, $P(V_{124})$, $P(V_{134})$, $P(V_{234})$, can be calculated. The final event probability to be computed in this example is $P(V_{1234})$, i.e., the probability that all signals in the maximal coverage set are usable. Based on the previous procedure, the probability is given by

$$\begin{aligned} P(V_{1234}) &= P\left[\ (A_1T_1)(A_2T_2)(A_3T_3)(A_4T_4)\ \right] = P\left[\ (A_1A_2A_3A_4)(T_1T_2T_3T_4)\ \right] \\ &= P(A_1A_2A_3A_4)P(T_1T_2T_3T_4) \end{aligned}$$

Thus, this probability is computed with the aid of Eqs. B.4-6, B.4-8, and B.4-3.

For the general case of m signals in the maximal coverage set, the expression for the probability of a usable signal subset event for an arbitrary number of signals is very complex. To simplify the analytic form, an expression for the general event will be given in terms of a sum (union) of mutually-exclusive events. Because of the exclusive event sum and the independence of the A- and T-events, the probability of the general event is readily obtained as in the

examples above. In order to obtain an expression for the event $V_{i_1 i_2 \ldots i_q}$ that the signals $i_1, i_2, \ldots i_q$ are usable within a maximal coverage set composed of signals $i_1, i_2, i_q, i_{q+1}, \ldots i_m$, the events Z_i, Y_i, and W are first defined, viz.,

$$Z_i = A_i T_i \; ; \quad Y_i = \overline{A}_i T_i = T_i - Z_i \; ; \quad W = Z_{i_1} Z_{i_2} \ldots Z_{i_q}$$

In terms of these (and previously defined) events, the event $V_{i_1 i_2 \ldots i_q}$ is expressed as

$$V_{i_1 i_2 \ldots i_q} = W \prod_{k=q+1}^{m} Y_{i_k} + W \sum_{j=q+1}^{m} \overline{T}_{i_j} \prod_{\substack{k=q+1 \\ k \neq j}}^{m} Y_{i_k} + W \sum_{j_1=q+1}^{m-1} \sum_{j_2=j_1+1}^{m} \overline{T}_{i_{j_1}} \overline{T}_{i_{j_2}} \prod_{\substack{k=q+1 \\ k \neq j_1; k \neq j_2}}^{m} Y_{i_k}$$

$$+ \ldots + W \sum_{j_1=q+1}^{m-r+1} \sum_{j_2=j_1+1}^{m-r+2} \cdots \sum_{j_r=j_{r-1}+1}^{m} \overline{T}_{i_{j_1}} \overline{T}_{i_{j_2}} \cdots \overline{T}_{i_{j_r}} \prod_{\substack{k=q+1 \\ k \neq j_1, j_2, \ldots j_r}}^{m} Y_{i_k}$$

$$+ \ldots + W \prod_{k=q+1}^{m} \overline{T}_{i_k} \qquad \text{(B.4-9)}$$

For purposes of determining computational requirements, it is instructive to determine the number of terms in the calculation of P(V) for various values of m (number of signals in the maximal coverage set). If a "term" is defined as the probability of an intersection of several events (e.g., Z_i, Y_i, as defined above), then for m=3 only one term needs to be computed. For m=4,

$$2 \binom{4}{3} + \binom{4}{4} = 9$$

terms are necessary. In the case of 5 signals in the maximal coverage set, the number of terms to be computed are

$$2^2 \binom{5}{3} + 2 \binom{5}{4} + \binom{5}{5} = 51$$

As can be seen, the number of terms rises rapidly with increasing m. The results are summarized in Table B.4-1 for all possible values of m. The number of terms shown corresponds essentially to the number of terms of Eq. B.3-1.

Table B.4-1 Number of Terms Required for Calculation of P(V) for all Subsets of the Maximal Coverage Set of m Signals, for Various Values of m

NUMBER OF SIGNALS (m) IN THE MAXIMAL COVERAGE SET	NUMBER OF TERMS REQUIRED FOR CALCULATION
3	1
4	9
5	51
6	233
7	939
8	3489

B.5 PHASE ERROR SUB-MODEL

The previous two sections were primarily concerned with the calculation of $P(V_{i_1 i_2 \ldots\ldots})$ as it appears in Eq. B.3-1. The other factor appearing in each term of Eq. B.3-1 is $P(C_i(R_T) | V_{i_1 i_2 \ldots\ldots})$. Calculation of these quantities will be of concern in this and the following section.

Phase errors, which are the principal source of navigation/position error, are assumed to be distributed according to a normal distribution with mean given by the phase bias, or prediction, error and standard deviation given by the RMS of the day-to-day phase variation (for a given hour) about the phase bias error. The phase bias and random error components are a function of hour/month/path. Currently, no adequate model of phase error as a function of spatial location exists so that the phase error components must be extracted from measurement data at monitor sites. This limits the spatial applicability of the model to the immediate vicinity of the monitor sites.

The phase error on paths from each of the transmitting stations to each of the monitor sites generally subtend large relative angles and are thus assumed independent. For a given

monitor site/time with m signals in the maximal coverage set, the joint phase error probability density function has the form

$$p_{\Delta\phi_1\Delta\phi_2\ldots\Delta\phi_m}(\delta\phi_1, \delta\phi_2, \ldots, \delta\phi_m) = \prod_{i=1}^{m} \frac{e^{-(\delta\phi_i - \overline{\delta\phi_i})^2/2\sigma_i^2}}{\sigma_i\sqrt{2\pi}} \tag{B.5-1}$$

where $\overline{\delta\phi_i}$ is the phase bias error and σ_i is the day-to-day variation for signal i. In general, $\overline{\delta\phi_i}$ and σ_i are functions of time (hour/month). Equation B.5-1 may be written in the more compact form

$$p_{\Delta\phi}(\delta\phi) = \sqrt{\frac{\det U}{(2\pi)^m}}\; e^{-1/2(\delta\phi - \overline{\delta\phi})^T U (\delta\phi - \overline{\delta\phi})} \tag{B.5-2}$$

where

$$\Delta\phi = \begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \cdot \\ \cdot \\ \Delta\phi_m \end{bmatrix} ; \; \delta\phi = \begin{bmatrix} \delta\phi_1 \\ \delta\phi_2 \\ \cdot \\ \cdot \\ \delta\phi_m \end{bmatrix} ; \; \overline{\delta\phi} = \begin{bmatrix} \overline{\delta\phi_1} \\ \overline{\delta\phi_2} \\ \cdot \\ \cdot \\ \overline{\delta\phi_m} \end{bmatrix} \tag{B.5-2a}$$

$$U = \begin{bmatrix} 1/\sigma_1{}^2 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1/\sigma_2{}^2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & & \ddots & \cdots & \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & & \cdots & & 1/\sigma_m{}^2 \end{bmatrix} ; \tag{B.5-2b}$$

T indicates vector or matrix transpose; and det means determinant.

B.6 POSITION CHANGE ESTIMATION SUB-MODEL

To convert Omega signal phase measurements to spatial position, a generic procedure is assumed which is believed similar to algorithms used in the navigation filters of most modern airborne Omega receivers. The procedure is classified as a *navigation* technique (instead of a position-fixing scheme) in which the previous position is known and the new position is incrementally updated. For the procedure described by this sub-model, updates are assumed to be

frequent enough (every few minutes) to track and identify lanes but also of sufficient spatial separation to permit more-or-less independent update measurements. With an assumed initial correct position and an oscillator/clock which has less than one microsecond drift between updates, only two signals are needed. But operational receivers do not always have correct initial position so that the clock offset calculation is needed and a minimum of three station signals is required. The essential elements of this sub-model are as follows:

(1) A least-squares estimate of two-dimensional position is made based on three or more phase measurements

(2) The phase measurements from which position is derived/estimated are unweighted

(3) The initial position is assumed correct so that error is introduced only in one update cycle

(4) Neither the phase measurements or intermediate position estimates are filtered

(5) The receiver clock/oscillator is assumed synchronized to "Omega time" at the beginning of an update cycle so that error due to clock drift (offset) occurs in one update cycle.

Elements (2), (3), and (4) above are necessary in order that the sub-model be free of history dependence, i.e., the error at a point in space/time should not depend on how the navigator arrived at the point. Although elements (3) and (5) are optimistic (compared to an operational receiver) in the sense that there is no cumulative error, they are partially compensated by elements (2) and (4) since the absence of signal weighting and filtering normally lead to increased errors.

In a state space formulation, the state variables are the two (north and east) components of the position change (between updates) and the receiver's clock/oscillator drift (due to frequency offset) between updates. The measurement variables are the corresponding phase changes between updates and the measurement process has its own noise/uncertainty. The position changes are assumed small enough so that a flat-earth approximation is valid. Thus the measurement equation is

$$\Delta\phi = H \, \Delta X' + v \qquad \text{(B.6-1)}$$

where

$$\Delta\phi = \begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \vdots \\ \Delta\phi_q \end{bmatrix} \; ; \; \Delta X' = \begin{bmatrix} \Delta N \\ \Delta E \\ \Gamma \end{bmatrix} \qquad \text{(B.6-1a)}$$

and

$$H = \begin{bmatrix} k_o \cos\beta_1 & k_o \sin\beta_1 & \tau \\ k_o \cos\beta_2 & k_o \sin\beta_2 & \tau \\ \vdots & \vdots & \vdots \\ k_o \cos\beta_q & k_o \sin\beta_q & \tau \end{bmatrix} \quad ; \; q \leq m \qquad \text{(B.6-1b)}$$

In this formulation, q is the number of usable signals at the given point in space/time, m is the number of signals in the maximal coverage set (also at the given space/time point), $\Delta\phi_i$ is the phase change measurement for signal i, ΔN and ΔE are the north and east components, respectively, of the position change, Γ is the clock drift rate (phase/unit time) during the update cycle, k_o is the nominal wave number (in units of phase/distance), and τ is the update time interval. The station signals 1,2, ..., q are chosen for notational convenience and do not necessarily correspond to the conventional station numbering scheme. The quantity Γ is assumed constant over the update interval, thus implying a linear clock drift — certainly a safe assumption between 3-minute updates. The q-vector v is the zero-mean noise vector associated with the measurement process.

The least squares estimate of $\Delta X'$ based on a redundant set of phase measurements is that which minimizes the expected value of the sum of (unweighted) squares of the measurement noise vector components, i.e., $E(v^T v)$. The resulting estimate is

$$\Delta\hat{X}' = \left(H^T H\right)^{-1} H^T \Delta\phi \qquad \text{(B.6-2)}$$

where the superscript T indicates transpose.

The model characterized by Eqs. B.6-1 and B.6-2 describes the relationship between measured and estimated parameter (position, phase, etc.) *changes* occurring over an update cycle. Since these relationships are linear, the corresponding *errors* in these quantities (e.g., position error, phase error) are similarly related. Thus,

$$\delta\phi = H\delta X' \qquad \text{(B.6-3)}$$

$$\delta X' = \left(H^T H\right)^{-1} H^T \delta\phi \qquad \text{(B.6-4)}$$

where $\delta\phi$ and $\delta X'$ are defined analogously to $\Delta\phi$, $\Delta X'$ in Eq. B.6-1a. Note that the error in the measurement noise vector v is assumed to be zero. Since expectation is a linear operator, Eqs. B.6-3 and B.6-4 provide relationships between the phase error and position/clock drift bias errors, i.e.,

$$\overline{\delta\phi} = H\overline{\delta X'} \qquad \text{(B.6-5)}$$

$$\overline{\delta X'} = \left(H^T H\right)^{-1} H^T \overline{\delta\phi} \qquad \text{(B.6-6)}$$

The joint probability density function for the phase errors (Eq. B.5-2) is a function of the q phase errors, $\delta\phi_1, \delta\phi_2, \ldots, \delta\phi_q$. To find the corresponding probability density function for $\delta X'$, a probability transformation is required. However, such a transformation can only take place between spaces of equal dimension. Thus, to effect the transformation, the 3-vector, ΔX, must be supplemented by q–3 independent variables.

Fortunately, such a procedure is not required for linear transformations of normally distributed random variables. Even if the spaces on which the random variables are defined have unequal dimension, it is known (Ref. 10) that a linear transformation of normally distributed random variables yields transformed variables which are also normally distributed. For the case considered here, this result means that the linear relation between phase errors and position/ clock drift errors (Eq. B.6-3 or Eq. B.6-4) transforms the phase error normal distribution (Eq. B.5-2) for q phase error variables to the following probability density function for position and clock bias error:

$$p_{\Delta X'}(\delta X') = \sqrt{\frac{\det W}{(2\pi)^3}}\; e^{-(1/2)(\delta X' - \overline{\delta X'})^T W (\delta X' - \overline{\delta X'})} \qquad \text{(B.6-7)}$$

where

$$W = H^T U\, H \qquad \text{(B.6-8)}$$

To obtain the probability density function over the position error variables alone, the density function, Eq. B.6-7, must be integrated over the third component of $\delta X'$, i.e., the clock drift rate error, $\delta\Gamma$. The integration is facilitated by separating δT from the other two components of $\delta X'$, i.e.,

$$\delta X' = \begin{bmatrix} \delta N \\ \delta E \\ \delta\Gamma \end{bmatrix} = J_X \delta X + J_\Gamma \delta\Gamma \qquad \text{(B.6-9)}$$

where

$$J_X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \;;\; J_\Gamma = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(B.6-9a)}$$

and

$$\delta X = \begin{pmatrix} \delta N \\ \delta E \end{pmatrix} \tag{B.6-9b}$$

Further defining

$$\delta X' - \overline{\delta X'} = J_X \delta X_M + J_\Gamma \delta \Gamma_M \tag{B.6-10}$$

where

$$\delta X_M = \delta X - \overline{\delta X} \;\; ; \;\; \delta \Gamma_M = \delta \Gamma - \overline{\delta \Gamma} \tag{B.6-10a}$$

permits the argument (ARG) of the exponential in Eq. B.6-7 to be written

$$\text{ARG} = -(1/2)\text{EXP} \equiv -(1/2)\left(\delta X' - \overline{\delta X'}\right)^T W \left(\delta X' - \overline{\delta X'}\right) \tag{B.6-11}$$

in a form such that

$$\text{EXP} = (J_X \delta X_M + J_\Gamma \delta \Gamma_M)^T W (J_X \delta X_M + J_\Gamma \delta \Gamma_M)$$

Carrying out the indicated multiplication and noting that terms which are transposes of each other are equal (since the terms themselves are scalars) yields

$$\text{EXP} = \delta X_M^T L_{XX} \delta X_M + 2\, L_{\Gamma X} \delta X_M \delta \Gamma_M + L_{\Gamma\Gamma} (\delta \Gamma_M)^2 \tag{B.6-12}$$

where

$$L_{XX} = J_X^T W J_X \;\; ; \;\; L_{\Gamma X} = J_\Gamma^T W J_X \;\; ; \;\; L_{\Gamma\Gamma} = J_\Gamma^T W J_\Gamma \tag{B.6-12a}$$

Since $\delta \Gamma_M$ is a scalar in the expression for EXP, an ordinary completion of the square technique may be used to isolate the $\delta \Gamma_M$ dependence. Thus, adding and subtracting

$$\frac{(L_{\Gamma X} \delta X_M)^2}{L_{\Gamma\Gamma}} = L_{\Gamma\Gamma} \left(\frac{L_{\Gamma X} \delta X_M}{L_{\Gamma\Gamma}} \right)^2$$

$$= L_{\Gamma\Gamma} \frac{\delta X_M^T \, L_{\Gamma X}^T \, L_{\Gamma X} \, \delta X_M}{L_{\Gamma\Gamma}^2}$$

from Eq. B.6-12 (note from Eq. B.6-12a that $L_{\Gamma\Gamma}$ is a scalar) yields

$$\text{EXP} = L_{\Gamma\Gamma} \left[\left(\delta \Gamma_M + \frac{L_{\Gamma X} \delta X_M}{L_{\Gamma\Gamma}} \right)^2 + \frac{\delta X_M^T L_{XX} \delta X_M}{L_{\Gamma\Gamma}} - \frac{\delta X_M^T L_{\Gamma X}^T L_{\Gamma X} \delta X_M}{L_{\Gamma\Gamma}^2} \right]$$

Recalling the definition of EXP (Eq. B.6-11) and inserting the above form into the argument of the exponential in the density function (Eq. B.6-7) yields

$$p_{\Delta X'}(\delta X') = \sqrt{\frac{\det W}{(2\pi)^3}}\; e^{-\frac{1}{2}L_{\Gamma\Gamma}\left(\delta\Gamma_M + \frac{L_{\Gamma X}\delta X_M}{L_{\Gamma\Gamma}}\right)^2}\; e^{-(1/2)\delta X_M^T Q \delta X_M} \qquad \text{(B.6-13)}$$

where

$$Q = L_{XX} - \frac{L_{\Gamma X}^T L_{\Gamma X}}{L_{\Gamma\Gamma}} \qquad \text{(B.6-13a)}$$

Integrating Eq. B.6-13 over all $\delta\Gamma_M$ ($-\infty$ to $+\infty$) and using the normalization condition for a single-variate normal density function yields the probability density function for position error only, viz

$$p_{\Delta X}(\delta X) = \sqrt{\frac{\det W}{(2\pi)^3}} \cdot \frac{1}{\sqrt{L_{\Gamma\Gamma}}} \cdot \sqrt{2\pi}\; e^{-(1/2)\,\delta X_M^T Q \delta X_M}$$

$$= \frac{1}{2\pi}\sqrt{\frac{\det W}{L_{\Gamma\Gamma}}}\; e^{-(1/2)\,\delta X_M^T Q \delta X_M} \qquad \text{(B.6-14)}$$

To establish the normalization of this density function, it is necessary to show only that

$$\frac{\det W}{L_{\Gamma\Gamma}} = \det Q \qquad \text{(B.6-15)}$$

To prove this relationship, det W and det Q are separately analyzed. W, defined by Eq. B.6-8, is a symmetric 3x3 matrix which may be written in the block matrix form

$$W = \begin{pmatrix} W_I & W_{II} \\ W_{II}^T & W_{33} \end{pmatrix} \qquad \text{(B.6-16)}$$

where

$$W_I = \begin{pmatrix} W_{11} & W_{12} \\ W_{12} & W_{22} \end{pmatrix} \; ; \quad W_{II} = \begin{pmatrix} W_{13} \\ W_{23} \end{pmatrix} \qquad \text{(B.6-17)}$$

To find the determinant of a block matrix, a result from matrix theory (Ref. 13) is used which states that

$$\det \begin{pmatrix} A & D \\ C & B \end{pmatrix} = \det A \det \left(B - CA^{-1}D\right)$$

where A is an mxm matrix, B is nxn, C is nxm, and D is mxn. Applying this result to Eq. B.6-16 yields

$$\det W = \det W_I \det \left(W_{33} - W_{II}^T W_I^{-1} W_{II}\right)$$

or

$$\frac{\det W}{W_{33}} = \det W_I \det \left(1 - \frac{W_{II}^T W_I^{-1} W_{II}}{W_{33}} \right) \tag{B.6-18}$$

From Eq. B.6-13a, it is seen that det Q may be written

$$\det Q = \det \left[L_{XX} \left(\mathbb{1}_2 - \frac{L_{XX}^{-1} L_{rX}^T L_{rX}}{L_{rr}} \right) \right]$$

where $\mathbb{1}_2$ is the 2x2 identity matrix. Since det (AB) = det A det B, where A and B are square matrices, det Q is written

$$\det Q = \det L_{XX} \det \left(\mathbb{1}_2 - \frac{L_{XX}^{-1} L_{rX}^T L_{rX}}{L_{rr}} \right) \tag{B.6-19}$$

To reduce this form further, another result from matrix theory (Ref. 13) is needed. This result states that

$$\det \left(\mathbb{1}_n - AB\right) = \det \left(\mathbb{1}_m - BA\right)$$

where matrix A is nxm, matrix B is mxn, $\mathbb{1}_n$ is the nxn identity matrix, and $\mathbb{1}_m$ is the mxm identity matrix. Applying this result to Eq. B.6-19 yields

$$\det Q = \det L_{XX} \det \left(1 - \frac{L_{rX} L_{XX}^{-1} L_{rX}^T}{L_{rr}} \right) \tag{B.6-20}$$

From the definitions of matrices J_X and J_Γ (Eq. B.6-9a), it is easily shown by direct matrix multiplication that L_{XX}, $L_{\Gamma X}$, and $L_{\Gamma\Gamma}$, as defined in Eq. B.6-12, are expressed in terms of the components of W as

$$L_{XX} = W_I \; ; \; L_{\Gamma X} = W_{II}^T \; ; \; L_{\Gamma\Gamma} = W_{33} \qquad \text{(B.6-21)}$$

where W_I and W_{II} are defined by Eq. B.6-17. Substituting these results into the form for det Q (Eq. B.6-20) yields a form identical to the right-hand side of Eq. B.6-18. Thus

$$\det Q = \frac{\det W}{W_{33}} = \frac{\det W}{L_{\Gamma\Gamma}}$$

and the normalization condition (Eq. B.6-15) is proved.

Hence, the position error density function (Eq. B.6-14) may be written (using the definition in Eq. B.6-10a)

$$p_{\Delta X}(\delta X) = \frac{1}{2\pi} \sqrt{\det Q} \; e^{-\frac{1}{2}(\delta X - \overline{\delta X})^T Q (\delta X - \overline{\delta X})} \qquad \text{(B.6-22)}$$

From this position error density function, most measures of two-dimensional accuracy can be derived.

B.7 RADIAL ERROR DISTRIBUTION FUNCTION AND FINAL EXPRESSION FOR P_{SAA}

The position error density function (Eq. B.6-22) is transformed to polar coordinates (r, θ) using the transformation

$$\delta N = r \sin\theta \; ; \; \delta E = r \cos\theta$$

The Jacobian of the transformation is just r so that Eq. B.6-22 becomes

$$p_{R,\Theta}(r,\theta) = \frac{r}{2\pi} \sqrt{\det Q} \; e^{-\frac{1}{2}(\delta X(r,\theta) - \overline{\delta X})^T Q (\delta X(r,\theta) - \overline{\delta X})}$$

Integrating over the polar angle θ gives the radial error density function,

$$p_R(r) = \frac{r}{2\pi} \sqrt{\det Q} \int_0^{2\pi} d\theta \; e^{-\frac{1}{2}(\delta X(r,\theta) - \overline{\delta X})^T Q (\delta X(r,\theta) - \overline{\delta X})}$$

Now, if a threshold radial error, R_T, is selected, the radial error density function may be integrated from $r = 0$ (the true position) to $r = R_T$ to obtain the *position error distribution function*,

$$P(r \leq R_T) = \int_0^{R_T} p_R(r)\, dr$$

$$= \frac{\sqrt{\det Q}}{2\pi} \int_0^{R_T} r\, dr \int_0^{2\pi} d\theta\; e^{-\frac{1}{2}\left(\delta X(r,\theta) - \overline{\delta X}\right)^T Q\left(\delta X(r,\theta) - \overline{\delta X}\right)} \qquad \text{(B.7-1)}$$

This double integral, which effectively computes the "volume" of a cylinder under a Gaussian surface (with symmetry axis different from the cylinder's symmetry axis), cannot be evaluated analytically in closed form. Besides R_T, the parameters implicit in Eq. B.7-1 are carried by Q which is given by Eq. B.6-13a. Thus, Q carries the parameters k_o, σ_1, σ_2, ..., σ_q, β_1, β_2, ..., β_q. Also, from Eq. B.6-6, $\overline{\delta X}$ carries the phase bias errors $\overline{\delta\phi_1}$, $\overline{\delta\phi_2}$, ..., $\overline{\delta\phi_q}$ in addition to k_o, β_1, β_2, ..., β_q, and τ. For notational convenience, the q usable signals addressed in Section B.6 are labeled 1,2, . . . , q without reference to the usual numbering of Omega stations. Thus, the distribution function P defined by Eq. B.7-1 should be labeled with the set of signals i_1, i_2, ..., i_q. In this sense, then, the first factor of each term in Eq. B.3-1 may be identified with a distribution function given by Eq. B.7-1, i.e.,

$$P\left(C_i(R_T) \mid V_{i_1,i_2,\ldots,i_q}\right) = P_{i_1,i_2,\ldots,i_q}(r \leq R_T)$$

Using Eq. B.3-1, $P(C_i(R_T))$ may be expressed as

$$P(C_i(R_T)) = \sum_{q=3}^{m} \sum_{\{i_1,i_2,\ldots,i_q\}_m} P_{i_1,i_2,\ldots,i_q}(r \leq R_T)\, P\left(V_{i_1,i_2,\ldots,i_q}\right)$$

where $\{i_1, i_2, \ldots, i_q\}_m$ is the set of all combinations of q signals where q ranges from 3 to m (number of signals in the maximal coverage set). Now using the definition of $C_i(R_T)$ (see Section B.3) and Eq. B.2-1, it follows that

$$P(D_i(R_T)) = \frac{1}{N_u} \sum_{j=1}^{n_c} n_j\, P_{R_j} \sum_{q=3}^{m} \sum_{\{i_1,i_2,\ldots,i_q\}_m} P_{i_1,i_2,\ldots,i_q}(r \leq R_T)\, P\left(V_{i_1,i_2,\ldots,i_q}\right)$$

With the use of Eq. B.1-1, the final expression for P_{SAA} is given as

$$P_{SAA} = P(D(R_T)) = \sum_{i=1}^{444} w_i \left(\frac{1}{N_u}\right) \sum_{j=1}^{n_c} n_j P_{R_j} \sum_{q=3}^{m} \sum_{\{i_1,i_2,\ldots,i_q\}_m} P_{i_1,i_2,\ldots,i_q}(r \le R_T)\, P\left(V_{i_1,i_2,\ldots,i_q}\right) \tag{B.7-2}$$

To see the dependence of P_{SAA} on the various parameters, it is useful to review the parameters associated with each part of Eq. B.7-2. The normalized weights, w_i, came from the cell weighting matrix of the user regional priority sub-model. Since the model only applies to those cells containing monitor sites with single-station phase data, most of the weights will be zero (only about 10 w_i-values will be non-zero). The quantity n_j is the number of users in user class j ($j \le n_c$, the total number of user classes) and the total number of users is

$$N_u = \sum_{j=1}^{n_c} n_j$$

P_{R_j} is the receiver reliability/availability figure for user class j. For PACE, the workstation which implements the P_{SA} calculation, only one user class ($j = 1$) is considered and $P_{R_1} = 1$ so that

$$\frac{1}{N_u} \sum_{j=1}^{n_c} n_j P_{R_j} = 1$$

The probability, $P(V_{i_1 i_2 \ldots i_q})$, that only signals from stations $i_1, i_2, \ldots, i_q$ are usable (at a particular time) in cell i, is computed from the event $V_{i_1 i_2 \ldots i_q}$, which is expressed in Eq. B.4-9 in terms of a union of mutually exclusive component events. The component event probabilities are computed using the methods discussed in Section B.4 in terms of the scheduled and unscheduled event probabilities (function of month and station signal), the mean and standard deviation of the amplitudes for signals $i_1, i_2, \ldots, i_q$ (functions of month, hour, and cell), and the mean and standard deviation of the noise level (also a function of month, hour, and cell). Finally, the distribution function $P_{i_1 i_2 \ldots i_q}(r \le R_T)$ is given by Eq. B.7-1 which contains the threshold error, R_T, the position bias error vector, $\overline{\delta X}$, which depends on the phase bias errors on signals $i_1, i_2, \ldots, i_q$ (function of hour and month) for cell i, the free space wave number, k_o, the receiver update time interval, τ, and the bearing angles to stations $i_1, i_2, \ldots, i_q$, and matrix Q which depends on the standard deviation of the day-to-day phase variation for signals $i_1, i_2, \ldots, i_q$ at the monitor

site in cell i (function of hour and month), the bearing angles from cell i to stations $i_1, i_2, \ldots, i_q$, and the free space wave number, k_o (function of frequency).

B.8 FURTHER ANALYSIS OF MATRIX Q

The position error density function is, like all normal probability density functions, specified by two quantities:

(1) The bias error vector $\overline{\delta X}$ which specifies the displacement of the symmetry axis of the distribution

(2) The matrix Q whose inverse describes the spread or peakedness of the distribution.

In this section, attention will be focused on matrix Q (and its inverse) since measures of Q may be expressed in a dimensionless form, assuming all phase error standard deviations are equal, which permits relative comparison of error distribution "spread" under different conditions. Toward this end, the determinant and inverse of Q^{-1} will be calculated.

The determinant of Q is most easily obtained from Eq. B.6-20, noting the definition of L_{TT} in Eq. B.6-21. Thus,

$$\det Q = \frac{\det W}{W_{33}} \tag{B.8-1}$$

where W is given by Eq. B.6-8. From the definition of H (Eq. B.6-1b) and U (Eq. B.5-2b), a simple matrix multiplication shows that

$$W = \begin{bmatrix} k_o^2 \sum_{i=1}^{q} \frac{\cos^2\beta_i}{\sigma_i^{\,2}} & k_o^2 \sum_{i=1}^{q} \frac{\cos\beta_i \sin\beta_i}{\sigma_i^{\,2}} & \tau k_o \sum_{i=1}^{q} \frac{\cos\beta_i}{\sigma_i^{\,2}} \\ k_o^2 \sum_{i=1}^{q} \frac{\sin\beta_i \cos\beta_i}{\sigma_i^{\,2}} & k_o^2 \sum_{i=1}^{q} \frac{\sin^2\beta_i}{\sigma_i^{\,2}} & \tau k_o \sum_{i=1}^{q} \frac{\sin\beta_i}{\sigma_i^{\,2}} \\ \tau k_o \sum_{i=1}^{q} \frac{\cos\beta_i}{\sigma_i^{\,2}} & \tau k_o \sum_{i=1}^{q} \frac{\sin\beta_i}{\sigma_i^{\,2}} & \tau^2 \sum_{i=1}^{q} \frac{1}{\sigma_i^{\,2}} \end{bmatrix} \tag{B.8-2}$$

The determinant of W is evaluated by expansion of minors along the third row (working from right to left). Thus

$$\det W = \tau^2 \sum_{i=1}^{q} \frac{1}{\sigma_i{}^2} M_1 - \tau k_o \sum_{i=1}^{q} \frac{\sin\beta_i}{\sigma_i{}^2} M_2 + \tau k_o \sum_{i=1}^{q} \frac{\cos\beta_i}{\sigma_i{}^2} M_3 \qquad \text{(B.8-3)}$$

The quantity M_1 is calculated in a straightforward way as follows:

$$\begin{aligned}
M_1 &= W_{11}W_{22} - W_{12}^2 \\
&= k_o^4 \sum_{j=1}^{q} \sum_{k=1}^{q} \frac{1}{\sigma_j{}^2 \sigma_k{}^2} \left[\cos^2\beta_j \sin^2\beta_k - \cos\beta_j \sin\beta_j \; \cos\beta_k \sin\beta_k\right] \\
&= k_o^4 \sum_{j=1}^{q} \sum_{k=1}^{q} \frac{1}{\sigma_j{}^2 \sigma_k{}^2} \cos\beta_j \sin\beta_k \; \sin\left(\beta_k - \beta_j\right) \\
&= \frac{k_o^4}{2} \sum_{j=1}^{q} \sum_{k=1}^{q} \frac{1}{\sigma_j{}^2 \sigma_k{}^2} \left[\sin\left(\beta_k + \beta_j\right) + \sin\left(\beta_k - \beta_j\right)\right] \sin\left(\beta_k - \beta_j\right) \\
&= \frac{k_o^4}{2} \sum_{j=1}^{q} \sum_{k=1}^{q} \frac{1}{\sigma_j{}^2 \sigma_k{}^2} \left[\sin\left(\beta_k + \beta_j\right) \sin\left(\beta_k - \beta_j\right) + \sin^2\left(\beta_k - \beta_j\right)\right]
\end{aligned}$$

The first term in the summand sums to zero since it is odd in the interchange of indices k and j, i.e., for every term corresponding to (k, j), there is a negative term corresponding to (j, k) (terms vanish for k=j). Thus

$$M_1 = \frac{k_o^4}{2} \sum_{j=1}^{q} \sum_{k=1}^{q} \frac{1}{\sigma_j{}^2 \sigma_k{}^2} \sin^2\left(\beta_k - \beta_j\right) \qquad \text{(B.8-4)}$$

In a similar way, M_2 is calculated to yield

$$M_2 = \frac{\tau k_o^3}{2} \sum_{j=1}^{q} \sum_{k=1}^{q} \frac{1}{\sigma_j{}^2 \sigma_k{}^2} \sin\left(\beta_k - \beta_j\right) \left(\cos\beta_j - \cos\beta_k\right) \qquad \text{(B.8-5)}$$

and

$$M_3 = \frac{\tau k_o^3}{2} \sum_{j=1}^{q} \sum_{k=1}^{q} \frac{1}{\sigma_j{}^2 \sigma_k{}^2} \sin\left(\beta_k - \beta_j\right) \left(\sin\beta_j - \sin\beta_k\right) \qquad \text{(B.8-6)}$$

Substituting Eqs. B.8-4, B.8-5, and B.8-6 into Eq. B.8-3 yields

$$\det W = \frac{\tau^2 k_o^4}{2} \sum_{i=1}^{q} \sum_{j=1}^{q} \sum_{k=1}^{q} \frac{\sin(\beta_k - \beta_j)}{\sigma_i{}^2 \sigma_j{}^2 \sigma_k{}^2} \Big[\sin(\beta_k - \beta_j) - \sin\beta_i(\cos\beta_j - \cos\beta_k) + \cos\beta_i(\sin\beta_j - \sin\beta_k) \Big]$$

$$= \frac{\tau^2 k_o^4}{2} \sum_{i=1}^{q} \sum_{j=1}^{q} \sum_{k=1}^{q} \frac{\sin(\beta_k - \beta_j)}{\sigma_i{}^2 \sigma_j{}^2 \sigma_k{}^2} \Big[\sin(\beta_k - \beta_j) + \sin(\beta_j - \beta_i) + \sin(\beta_i - \beta_k) \Big] \tag{B.8-7}$$

The summand factor in brackets has the property that the sum of the arguments of the sines vanishes. In general, it can be shown that

$$\sin a + \sin\beta + \sin(-(a + \beta)) = \sin a + \sin\beta - \sin(a + \beta)$$
$$= 4 \sin(a/2)\sin(\beta/2)\sin\left({}^{(a+\beta)}/2\right) \tag{B.8-8}$$

Applying this result to Eq. B.8-7 yields

$$\det W = 2\, \tau^2 k_o^4 \sum_{i=1}^{q} \sum_{j=1}^{q} \sum_{k=1}^{q} \sin(\beta_k - \beta_j) \left[\frac{\sin(\frac{\beta_k - \beta_j}{2}) \sin(\frac{\beta_j - \beta_i}{2}) \sin(\frac{\beta_i - \beta_k}{2})}{\sigma_i{}^2 \sigma_j{}^2 \sigma_k{}^2} \right] \tag{B.8-9}$$

In this sum, terms which have any pair of indices equal vanish so that equal indices may be excluded from the sum. For any given triple of indices, say i', j', k', there are six possible permutations in which no two indices are equal. These permutations can be divided into cyclical permutations and pairwise permutations as follows:

Unpermuted: (i', j', k')

Cyclically permuted: (j', k', i'), (k', i', j')

Pairwise permuted: (j', i', k'), (k', j', i'), (i', k', j').

The summand factor in brackets in Eq. B.8-9 is symmetric under cyclical permutations (positive parity) and antisymmetric under pairwise permutation (negative parity). Thus, for a unique combination (triple) of indices, the following six values of the summand in Eq. B.8-9 are obtained:

Unpermuted: (i', j', k') : $B \sin\left(\beta_{k'} - \beta_{j'}\right)$

$$
\begin{array}{lll}
\text{Cyclical: } (j', k', i') & : & B \sin\left(\beta_{i'} - \beta_{k'}\right) \\
(k', i', j') & : & B \sin\left(\beta_{j'} - \beta_{i'}\right) \\
\text{Pairwise: } (j', i', k') & : - & B \sin\left(\beta_{k'} - \beta_{i'}\right) \\
(i', k', j') & : - & B \sin\left(\beta_{j'} - \beta_{k'}\right) \\
(k', j', i') & : - & B \sin\left(\beta_{i'} - \beta_{j'}\right).
\end{array}
$$

where B is the summand factor in brackets in Eq. B.8-9. Thus, if the sum in Eq. B.8-9 is taken over unique *combinations* of indices i, j, k, then the summand is composed of the six quantities given above, i.e.,

$$2\ B\left(\sin(\beta_{k'} - \beta_{j'}) + \sin(\beta_{i'} - \beta_{k'}) + \sin(\beta_{j'} - \beta_{i'})\right)$$

The arguments of the three sine terms sum to zero and thus the sum is equivalent to the form given in Eq. B.8-8. The summand thus becomes, on removing primes from the indices,

$$\frac{8\ \sin^2\left(\frac{\beta_k - \beta_j}{2}\right)\sin^2\left(\frac{\beta_i - \beta_k}{2}\right)\sin^2\left(\frac{\beta_j - \beta_i}{2}\right)}{\sigma_i^{\ 2}\sigma_j^{\ 2}\sigma_k^{\ 2}}$$

Writing the sum in Eq. B.8-9 over only unique combinations of indices and using the above form for the summand yields

$$\det W = 16\ \tau^2 k_o^4 \sum_{i=1}^{q-2}\ \sum_{j=i+1}^{q-1}\ \sum_{k=j+1}^{q} \frac{\sin^2\left(\frac{\beta_k - \beta_j}{2}\right)\sin^2\left(\frac{\beta_i - \beta_k}{2}\right)\sin^2\left(\frac{\beta_j - \beta_i}{2}\right)}{\sigma_i^{\ 2}\sigma_j^{\ 2}\sigma_k^{\ 2}}$$

Substituting this expression for det W into the equation for det Q (Eq. B.8-1) and noting, from Eq. B.8-2, that

$$W_{33} = \tau^2 \sum_{i=1}^{q} \frac{1}{\sigma_i^{\ 2}} \quad ,$$

the following form is obtained for det Q

$$\det Q = \frac{16\ k_o^4}{\sum_{i=1}^{q} \frac{1}{\sigma_i^{\ 2}}} \sum_{i=1}^{q-2}\ \sum_{j=i+1}^{q-1}\ \sum_{k=j+1}^{q} \frac{\sin^2\left(\frac{\beta_k - \beta_j}{2}\right)\sin^2\left(\frac{\beta_i - \beta_k}{2}\right)\sin^2\left(\frac{\beta_j - \beta_i}{2}\right)}{\sigma_i^{\ 2}\sigma_j^{\ 2}\sigma_k^{\ 2}} \qquad \text{(B.8-10)}$$

It is seen that the square root of this quantity, which carries the dimensions of the density function (see Eq. B.6-22) has dimensions of $(\text{distance})^{-2}$, which is correct for a two-dimensional density function defined over distance.

For the case of three usable signals (q=3), only one combination (1,2,3) of indices occurs and Eq. B.8-10 becomes

$$\det Q = \frac{16\ k_o^4}{\left(\frac{1}{\sigma_1{}^2} + \frac{1}{\sigma_2{}^2} + \frac{1}{\sigma_3{}^2}\right)} \cdot \frac{\sin^2(\frac{\beta_1-\beta_2}{2})\sin^2(\frac{\beta_2-\beta_3}{2})\sin^2(\frac{\beta_3-\beta_1}{2})}{\sigma_1{}^2\sigma_2{}^2\sigma_3{}^2}$$

For four usable signals, the sum comprises four terms, for q=5, ten terms are obtained, and, in general, for q usable signals,

$$\binom{q}{3}$$

terms occur in the sum for det Q.

From the discussion above, it is seen that

$$\sqrt{\det Q^{-1}} = \frac{1}{\sqrt{\det Q}}$$

has the dimensions of (radial distance error)^2. It follows, then, that $(\det Q)^{-4}$ is a linear measure of error. Computation of this error measure requires a knowledge of phase errors on all usable signals and bearing angles to all stations (from the receiver) transmitting usable signals. The bearing angles (β_i) are easily computed but the phase errors (σ_i) are known only at those monitor sites which measure single station phase. Thus, it is useful to consider an error measure (based on a highly idealized model) which is independent of phase error.

If all phase error standard deviations are assumed equal, i.e.,

$$\sigma_i = \sigma \quad , \quad i = 1,2,\ldots.\ q$$

then det Q (Eq. B.8-10) becomes

$$\det Q = \frac{16}{q}\left(\frac{k_o}{\sigma}\right)^4 \sum_{i=1}^{q-2}\ \sum_{j=i+1}^{q-1}\ \sum_{k=j+1}^{q} \sin^2\left(\frac{\beta_k-\beta_j}{2}\right)\sin^2\left(\frac{\beta_i-\beta_k}{2}\right)\sin^2\left(\frac{\beta_j-\beta_i}{2}\right)$$

As discussed above, a linear error measure, S, is obtained as the inverse fourth root, i.e.,

$$S = \frac{1}{(\det Q)^{1/4}} = \frac{\sigma q^{1/4}}{2k_o}\left[\sum_{i=1}^{q-2}\ \sum_{j=i+1}^{q-1}\ \sum_{k=j+1}^{q} \sin^2\left(\frac{\beta_k-\beta_j}{2}\right)\sin^2\left(\frac{\beta_i-\beta_k}{2}\right)\sin^2\left(\frac{\beta_j-\beta_i}{2}\right)\right]^{-\frac{1}{4}}$$

Finally, a dimensionless form, S′, is obtained as follows:

$$S' = \frac{k_o S}{\sigma} = \frac{q^{1/4}}{2}\left[\sum_{i=1}^{q-2}\sum_{j=i+1}^{q-1}\sum_{k=j+1}^{q}\sin^2\left(\frac{\beta_k-\beta_j}{2}\right)\sin^2\left(\frac{\beta_i-\beta_k}{2}\right)\sin^2\left(\frac{\beta_j-\beta_i}{2}\right)\right]^{-\frac{1}{4}} \tag{B.8-11}$$

This "GDOP-type" error contains only the bearing angles to the stations transmitting usable signals and, though based on a highly simplified model, S′ may be used to compare the relative accuracy of different usable signal sets at the same location or the same signal set at different locations.

The trace of Q^{-1} is obtained by first noting that, for the 2x2 matrix Q,

$$\mathrm{Tr}\ Q^{-1} = \frac{\mathrm{Tr}\ Q}{\det\ Q} \tag{B.8-12}$$

Since the trace is a linear operator, Tr Q may be expressed as the difference of two traces using Eq. B.6-13a as an expression for Q, i.e.,

$$\mathrm{Tr}\ Q = \mathrm{Tr}\ L_{XX} - \frac{\mathrm{Tr} L_{TX}^T L_{TX}}{L_{TT}} \tag{B.8-13}$$

Using definitions given by Eqs. B.6-17 and B.6-21, it is seen that

$$L_{XX} = W_I = \begin{pmatrix} W_{11} & W_{12} \\ W_{12} & W_{22} \end{pmatrix}$$

$$L_{TX}^T L_{TX} = W_{II}\ W_{II}^T = \begin{pmatrix} W_{13}^2 & W_{13}W_{23} \\ W_{13}W_{23} & W_{23}^2 \end{pmatrix}$$

$$L_{TT} = W_{33}$$

Thus, from Eq. B.8-13, Tr Q becomes

$$\mathrm{Tr}\ Q = W_{11} + W_{22} - \frac{\left(W_{13}^2 + W_{23}^2\right)}{W_{33}} \tag{B.8-14}$$

Now, substituting for the components of W, using Eq. B.8-2, yields

$$W_{11} + W_{22} = k_o^2 \sum_{i=1}^{q} \frac{1}{{\sigma_i}^2}$$

and

$$W_{13}^2 + W_{23}^2 = \tau^2 k_o^2 \sum_{i=1}^{q} \sum_{j=1}^{q} \frac{1}{\sigma_i{}^2 \sigma_j{}^2} \left(\cos\beta_i \cos\beta_j + \sin\beta_i \sin\beta_j\right)$$

$$= \tau^2 k_o^2 \sum_{i=1}^{q} \sum_{j=1}^{q} \frac{\cos(\beta_i - \beta_j)}{\sigma_i{}^2 \sigma_j{}^2}$$

With these results, Tr Q (Eq. B.8-14) becomes

$$\mathrm{Tr}\ Q = k_o^2 \sum_{i=1}^{q} \frac{1}{\sigma_i{}^2} - \frac{\tau^2 k_o^2}{\tau^2 \sum_{i=1}^{q} \frac{1}{\sigma_i{}^2}} \sum_{i=1}^{q} \sum_{j=1}^{q} \frac{\cos(\beta_i - \beta_j)}{\sigma_i{}^2 \sigma_j{}^2}$$

$$= \frac{k_o^2}{\sum_{i=1}^{q} \frac{1}{\sigma_i{}^2}} \sum_{i=1}^{q} \sum_{j=1}^{q} \frac{1}{\sigma_i{}^2 \sigma_j{}^2} \left(1 - \cos(\beta_i - \beta_j)\right)$$

$$= \frac{2k_o^2}{\sum_{i=1}^{q} \frac{1}{\sigma_i{}^2}} \sum_{i=1}^{q} \sum_{j=1}^{q} \frac{\sin^2\left(\frac{\beta_i - \beta_j}{2}\right)}{\sigma_i{}^2 \sigma_j{}^2}$$

Using this expression for Tr Q, Eq. B.8-12 gives the trace of Q inverse as

$$\mathrm{Tr}\ Q^{-1} = \frac{2k_o^2 \sum_{i=1}^{q} \sum_{j=1}^{q} \frac{\sin^2\left(\frac{\beta_i - \beta_j}{2}\right)}{\sigma_i{}^2 \sigma_j{}^2}}{\det Q \sum_{i=1}^{q} \frac{1}{\sigma_i{}^2}} \tag{B.8-15}$$

where det Q is given by Eq. B.8-10. Since the summand in the numerator of Eq. B.8-15 is symmetric under interchange of i and j, the sum may be written over unique combinations of i and j

(with a factor of two). Using this fact and substituting for det Q from Eq. B.8-10 into Eq. B.8-15 yields

$$\mathrm{Tr}\ Q^{-1} = \frac{\displaystyle\sum_{i=1}^{q-1} \sum_{j=i+1}^{q} \frac{\sin^2\left(\frac{\beta_i-\beta_j}{2}\right)}{\sigma_i{}^2\sigma_j{}^2}}{\displaystyle 4k_o^2 \sum_{i=1}^{q-2} \sum_{j=i+1}^{q-1} \sum_{k=j+1}^{q} \frac{\sin^2\left(\frac{\beta_k-\beta_j}{2}\right)\sin^2\left(\frac{\beta_i-\beta_k}{2}\right)\sin^2\left(\frac{\beta_j-\beta_i}{2}\right)}{\sigma_i{}^2\sigma_j{}^2\sigma_k{}^2}} \qquad \text{(B.8-16)}$$

Note that this quantity has dimensions of (distance)^2 which is correct, since Q has dimensions of (distance)$^{-2}$.

For three usable signals (q=3), Eq. B.8-16 reduces to

$$\mathrm{Tr}\ Q^{-1} = \frac{\dfrac{\sin^2\left(\frac{\beta_1-\beta_2}{2}\right)}{\sigma_1{}^2\sigma_2{}^2} + \dfrac{\sin^2\left(\frac{\beta_2-\beta_3}{2}\right)}{\sigma_2{}^2\sigma_3{}^2} + \dfrac{\sin^2\left(\frac{\beta_1-\beta_3}{2}\right)}{\sigma_1{}^2\sigma_3{}^2}}{4k_o^2\,\dfrac{\sin^2\left(\frac{\beta_1-\beta_2}{2}\right)\sin^2\left(\frac{\beta_2-\beta_3}{2}\right)\sin^2\left(\frac{\beta_1-\beta_3}{2}\right)}{\sigma_1{}^2\sigma_2{}^2\sigma_3{}^2}}$$

$$= \frac{1}{4k_o^2}\left[\frac{\sigma_3{}^2}{\sin^2\left(\frac{\beta_2-\beta_3}{2}\right)\sin^2\left(\frac{\beta_1-\beta_3}{2}\right)} + \frac{\sigma_1{}^2}{\sin^2\left(\frac{\beta_1-\beta_2}{2}\right)\sin^2\left(\frac{\beta_1-\beta_3}{2}\right)} + \frac{\sigma_2{}^2}{\sin^2\left(\frac{\beta_1-\beta_2}{2}\right)\sin^2\left(\frac{\beta_2-\beta_3}{2}\right)}\right]$$

When the bearing angle difference between a given station and any other station (in the usable coverage set) is small (or near 2π), then this error measure is more sensitive to bearing angle difference changes than to the phase error variance on the given station signal.

A linear error measure is obtained by taking

$$V = \sqrt{\mathrm{Tr}\ Q^{-1}}$$

As in the case of error measure S, described above, V is a function of the phase errors, σ_i, for the usable signals i=1,2, ..., q, as well as the bearing angles, β_i, i=1,2, ..., q. As discussed above, the phase error standard deviations, σ_i, are obtained only by measurement at monitor sites and thus V is severely limited in spatial extent as an error measure. For this reason, a simplified

model is assumed in which all phase error standard deviations are equal. In this case, V becomes, with the use of Eq. B.8-16,

$$V = \frac{\sigma}{2k_o}\left[\frac{\sum_{i=1}^{q-1}\sum_{j=i+1}^{q}\sin^2\left(\frac{\beta_i-\beta_j}{2}\right)}{\sum_{i=1}^{q-2}\sum_{j=i+1}^{q-1}\sum_{k=j+1}^{q}\sin^2\left(\frac{\beta_k-\beta_j}{2}\right)\sin^2\left(\frac{\beta_i-\beta_k}{2}\right)\sin^2\left(\frac{\beta_j-\beta_i}{2}\right)}\right]^{1/2} \qquad \text{(B.8-17)}$$

As in the case for det Q above, a dimensionless form is obtained by taking

$$V' = \frac{k_o}{\sigma} V$$

Thus, using Eq. B.8-17, V′ is given by

$$V' = \frac{1}{2}\left[\frac{\sum_{i=1}^{q-1}\sum_{j=i+1}^{q}\sin^2\left(\frac{\beta_i-\beta_j}{2}\right)}{\sum_{i=1}^{q-2}\sum_{j=i+1}^{q-1}\sum_{k=j+1}^{q}\sin^2\left(\frac{\beta_k-\beta_j}{2}\right)\sin^2\left(\frac{\beta_i-\beta_k}{2}\right)\sin^2\left(\frac{\beta_j-\beta_i}{2}\right)}\right]^{1/2}$$

This form is independent of phase error standard deviation and depends only on bearing angle differences. Note that $V' \rightarrow \infty$ when $\beta_i \rightarrow \beta_j$ for any i,j=1,2, ..., q. This situation occurs when the receiver lies on the baseline extension for two of the stations transmitting usable signals. Since Q^{-1} is sometimes referred to as the covariance matrix, V′ is a measure of the rms of the north and east error components.

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